

THE CATEGORY OF ORDERED BRATTELI DIAGRAMS

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ABSTRACT. A category structure for ordered Bratteli diagrams is proposed such that isomorphism in this category coincides with Herman, Putnam, and Skau's notion of equivalence. It is shown that the one-to-one correspondence between the category of essentially minimal totally disconnected dynamical systems and the category of essentially simple ordered Bratteli diagrams at the level of objects is in fact an equivalence of categories. In particular, we show that the category of Cantor minimal systems is equivalent to the category of properly ordered Bratteli diagrams. We obtain a model (diagram) for a homomorphism between essentially minimal totally disconnected dynamical systems, which may be useful in the study of factors and extensions of such systems. Various functors between the categories of essentially minimal totally disconnected dynamical systems, ordered Bratteli diagrams, AF algebras, and dimension groups are constructed and used to study the relations between these categories.

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1. INTRODUCTION

In 1972, Bratteli in a seminal paper introduced what are now called Bratteli diagrams to study AF algebras [2]. He associated to each AF algebra an infinite directed graph, its Bratteli diagram (see Definition 2.1), and used these

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very effectively to study AF algebras. Some attributes of an AF algebra can be read directly from its Bratteli diagram (e.g., ideal structure).

Based on the notion of a Bratteli diagram, Elliott introduced dimension groups and gave a classification of AF algebras using K-theory in 1976 [7]. In fact, Elliott showed that the functor $K_0 : \mathbf{AF} \rightarrow \mathbf{DG}$, from the category of AF algebras with $*$ -homomorphisms to the category of dimension groups with order-preserving homomorphisms, is a strong classification functor (see [7] and [8, Sections 5.1–5.3]). Recall that a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ was called in [8] a classification functor if $F(a) \cong F(b)$ implies $a \cong b$, for each $a, b \in \mathcal{C}$, and a strong classification functor if each isomorphism from $F(a)$ to $F(b)$ is the image of an isomorphism from a to b .

In [1], the authors introduced a notion of morphism between Bratteli diagrams such that, in the resulting category of Bratteli diagrams, \mathbf{BD} , isomorphism of Bratteli diagrams coincides with Bratteli's notion of equivalence. We showed that the map $\mathcal{B} : \mathbf{AF} \rightarrow \mathbf{BD}$, defined by Bratteli in [2] on objects, is in fact a functor. The fact that this is a strong classification functor, [1, Theorem 3.11], is a functorial formulation of Bratteli's classification of AF algebras and completes his work from the classification point of view introduced by Elliott in [8].

In a notable development, Bratteli diagrams have been used to study certain dynamical systems. In 1981, A. V. Vershik used Bratteli diagrams to construct so-called adic transformations [18, 19]. Based on his work (and the work of S. C. Power [16]), Herman, Putnam, and Skau introduced the notion of an ordered Bratteli diagram and associated a dynamical system to each (essentially simple) ordered Bratteli diagram [12]. In fact, they showed that there is a one-to-one correspondence between essentially simple ordered Bratteli diagrams and essentially minimal totally disconnected dynamical systems [12, Theorem 4.7]. (In particular, each Cantor minimal system has a Bratteli-Vershik model.) This correspondence was used effectively to study Cantor minimal dynamical systems and the characterization of various types of orbit equivalence in terms of isomorphism of the related C^* -algebra crossed products and dimension groups [12, 9, 10, 14].

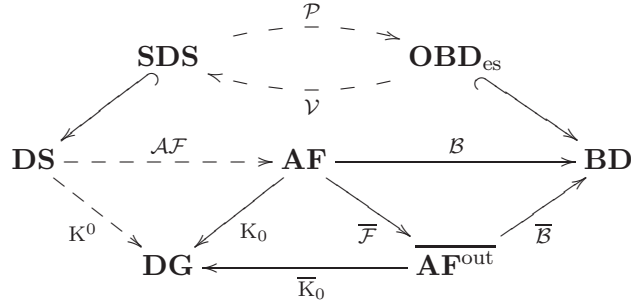
In this paper, we propose a notion of (ordered) morphism between ordered Bratteli diagrams and obtain the *category* of ordered Bratteli diagrams, \mathbf{OBD} (Theorem 3.5). Isomorphism in this category coincides with the notion of equivalence in the sense of Herman, Putnam, and Skau (Theorem 3.6). We show that the correspondence obtained by Herman, Putnam, and Skau in [12] (mentioned above) is an equivalence of categories. Denote by \mathbf{SDS} the category of scaled essentially minimal totally disconnected dynamical systems (see Definition 4.5). We construct a contravariant functor $\mathcal{P} : \mathbf{SDS} \rightarrow \mathbf{OBD}$, thus obtaining what might be viewed as a model (diagram) for a homomorphism between essentially minimal totally disconnected dynamical systems. (In particular, this may be useful in the study of factors and extensions of such systems.)

In Section 4, we show that the contravariant functor $\mathcal{P} : \mathbf{SDS} \rightarrow \mathbf{OBD}$ is full and faithful. The fact that this functor is full is a tool to obtain homomorphisms between the dynamical systems in question by graphically constructing certain arrows (i.e., morphisms) between the associated Bratteli diagrams. The (essential) range of this functor is the class of essentially simple ordered Bratteli diagrams \mathbf{OBD}_{es} . Hence $\mathcal{P} : \mathbf{SDS} \rightarrow \mathbf{OBD}_{\text{es}}$ is an equivalence of categories (Theorem 4.14).

In Section 5, we construct a functor inverse to the functor \mathcal{P} , a contravariant functor $\mathcal{V} : \mathbf{OBD}_{\text{es}} \rightarrow \mathbf{SDS}$ which is also an equivalence of categories. Our definition of this functor on objects uses ideas from [12]. We compute the correspondences τ and σ implementing the natural isomorphisms $1_{\mathbf{OBD}_{\text{es}}} \cong \mathcal{P}\mathcal{V}$ and $1_{\mathbf{SDS}} \cong \mathcal{V}\mathcal{P}$. (This is in particular useful when one wants to apply these functors to morphisms.) In this way, we obtain a functorial formulation of the correspondence of [12] between essentially simple ordered Bratteli diagrams and essentially minimal totally disconnected dynamical systems (Theorem 5.6).

In particular, in Section 6 we show that the category of minimal dynamical systems based on compact, totally disconnected metrizable spaces is equivalent to the category of simple ordered Bratteli diagrams (Corollary 6.3), and the category of minimal dynamical systems on the Cantor set is equivalent to the category of properly ordered Bratteli diagrams (Corollary 6.7).

The relations between AF algebras, Bratteli diagrams, dimension groups, and dynamical systems have been studied in various papers (for example, see [7, 17, 12, 9, 4, 1]). Most of the results concentrate on objects. We wish to give a functorial picture of relations between these categories (to encompass also morphisms). Thus, in this note we construct the missing functors in the following commutative (up to natural isomorphism) diagram:



Here, \mathbf{DS} , \mathbf{SDS} , \mathbf{DG} , \mathbf{BD} , \mathbf{OBD} , \mathbf{AF} , and $\overline{\mathbf{AF}^{\text{out}}}$ denote the categories of essentially minimal totally disconnected dynamical systems, scaled essentially minimal totally disconnected dynamical systems, dimension groups, Bratteli diagrams, ordered Bratteli diagrams, AF algebras, and the abstract category associated with AF algebras in [8], respectively. The precise definitions of these categories will be given below. (See [1] for the precise definition

of the categories \mathbf{DG} , \mathbf{BD} , \mathbf{AF} , and $\overline{\mathbf{AF}^{\text{out}}}$; the others are defined in the sequel.)

The functor $K_0 : \mathbf{AF} \rightarrow \mathbf{DG}$ was introduced by Elliott in [7] and is a strong classification functor (see also [8, 1]). The functors $\mathcal{B} : \mathbf{AF} \rightarrow \mathbf{BD}$ and $\overline{\mathcal{F}} : \mathbf{AF} \rightarrow \overline{\mathbf{AF}^{\text{out}}}$ are also strong classification functors [8, 1]. The functors $\overline{K}_0 : \overline{\mathbf{AF}^{\text{out}}} \rightarrow \mathbf{DG}$ and $\overline{\mathcal{B}} : \overline{\mathbf{AF}^{\text{out}}} \rightarrow \mathbf{BD}$ are equivalence of categories [1]. The functors $\mathcal{P} : \mathbf{SDS} \rightarrow \mathbf{OBD}_{\text{es}}$ and $\mathcal{V} : \mathbf{OBD}_{\text{es}} \rightarrow \mathbf{SDS}$ are also equivalence of categories (see Sections 4 and 5). The functor $K^0 : \mathbf{DS} \rightarrow \mathbf{DG}$ is defined in Section 8 and the functor $\mathcal{AF} : \mathbf{DS} \rightarrow \mathbf{AF}$ is defined in Section 9 and is a faithful functor. The direct limit functor $\mathcal{D} : \mathbf{BD} \rightarrow \mathbf{DG}$ (see Section 7) is an equivalence of categories.

2. PRELIMINARIES

There are various definitions of a Bratteli diagram which are essentially equivalent. The present work is a continuation of [1], using certain ideas from [17, 12, 9]. For the definition of a Bratteli diagram we shall follow [12], but the relation to the notions of [1] will be discussed.

Definition 2.1 ([12], Definition 2.1). A Bratteli diagram consists of a vertex set V and an edge set E satisfying the following conditions. We have a decomposition of V as a disjoint union $V_0 \cup V_1 \cup \dots$, where each V_n is finite and non-empty and V_0 has exactly one element, v_0 . Similarly, E decomposes as a disjoint union $E_1 \cup E_2 \cup \dots$, where each E_n is finite and non-empty. Moreover, we have maps $r, s : E \rightarrow V$ such that $r(E_n) \subseteq V_n$ and $s(E_n) \subseteq V_{n-1}$, $n = 1, 2, 3, \dots$ (r = range, s = source). We also assume that $s^{-1}\{v\}$ is non-empty for all v in V and $r^{-1}\{v\}$ is non-empty for all v in $V \setminus V_0$. Let us denote such a B by the diagram

$$V_0 \xrightarrow{E_1} V_1 \xrightarrow{E_2} V_2 \xrightarrow{E_3} \dots$$

Let us denote by \mathbf{BD}_1 the class of all Bratteli diagrams in the sense of Definition 2.1 (the superscript 1 will also appear later).

Definition 2.2 ([1], Definition 2.2). By a Bratteli diagram let us mean an ordered pair $B = (V, E)$, $V = (V_n)_{n=1}^\infty$ and $E = (E_n)_{n=1}^\infty$, such that:

- (1) each V_n is a $k_n \times 1$ matrix of non-zero positive integers for some $k_n \geq 1$;
- (2) each E_n is an embedding matrix from V_n to V_{n+1} , i.e., a $k_{n+1} \times k_n$ matrix of positive integers (including zero) with no zero columns, such that $E_n V_n \leq V_{n+1}$, in the coordinate-wise order.

As in [1], we shall use the notation \mathbf{BD} for the set of Bratteli diagrams in the sense of Definition 2.2. Denote by \mathbf{BD}^1 the set of all (V, E) in \mathbf{BD} such that each E_n has no zero column and that $E_n V_n = V_{n+1}$, for each $n \geq 1$. In fact, \mathbf{BD}^1 is the image of \mathbf{AF}_1 , the class of all unital AF algebras, under the functor $\mathcal{B} : \mathbf{AF} \rightarrow \mathbf{BD}$ introduced in [1] (by [1, Proposition 4.3]).

The essential difference between Definition 2.1 and Definition 2.2 is the specification of a total ordering on the vertices of each level. In fact, the categories \mathbf{BD}_1 and \mathbf{BD}^1 are equivalent (see Theorem 2.15 below). We will adopt Definition 2.1 in the sequel.

Let us recall some definitions and fix some notation concerning Bratteli diagrams [12, 9, 4, 1].

With the notation of Definition 2.1, if we fix a total order on each V_n then to each edge set E_n is associated a matrix called the *multiplicity matrix* of E_n , which we denote by $M(E_n)$ (it was called the “incidence matrix” in [9]; the expression “multiplicity matrix” was used in [1] for the matrix associated to a homomorphism between finite dimensional C^* -algebras). The number of columns and rows of $M(E_n)$ equals the number of elements of V_{n-1} and V_n , respectively, and each entry of the matrix gives the number of edges between two vertices.

Let l, k be integers with $0 \leq k < l$. Let $E_{k+1} \circ E_k \circ \cdots \circ E_l$, or also $E_{k,l}$, denote the set of all paths from V_k to V_l , i.e., the following set:

$$\{(e_{k+1}, \dots, e_l) \mid e_i \in E_i, i = k+1, \dots, l, r(e_i) = s(e_{i+1}), i = k+1, \dots, l\}.$$

We define $r(e_{k+1}, \dots, e_l) = r(e_l)$ and $s(e_{k+1}, \dots, e_l) = s(e_{k+1})$. Note that $M(E_{k,l}) = M(E_l) \cdots M(E_k)M(E_{k+1})$. Also set $E_{k,k} = \{(v, v) \mid v \in V_k\}$ which is an edge set from V_k to itself and its multiplicity matrix is the identity matrix of order equal to the cardinality of V_n .

Definition 2.3 ([12], Definition 2.2). Given a Bratteli diagram (V, E) and a sequence $m_0 = 0 < m_1 < m_2 < \cdots$ in \mathbb{N} , the *contraction (telescoping)* of (V, E) to $\{m_n\}_0^\infty$ is the Bratteli diagram (V', E') where $V'_n = V_{m_n}$, for $n \geq 0$, and $E'_n = E_{m_{n-1}, m_n}$, and $r' = r$, $s' = s$ as above.

Two Bratteli diagrams (V, E) and (V', E') were called *isomorphic* in [12] if there exist bijections between V and V' and between E and E' , preserving the gradings and intertwining the respective source and range maps. Also, Herman, Putnam, and Skau considered a notion of *equivalence* of Bratteli diagrams, denoted by \sim , in [12] (basically due to Bratteli in [2]). In fact, this is the equivalence relation generated by isomorphism (as in [12]) and telescoping. As indicated in [4], the Bratteli diagrams (V^1, E^1) and (V^2, E^2) are equivalent if and only if there exists a Bratteli diagram (V, E) such that the telescoping of (V, E) to odd levels $0 < 1 < 3 < \cdots$ yields a telescoping of (V^1, E^1) and the telescoping of (V, E) to even levels $0 < 2 < 4 < \cdots$ yields a telescoping of (V^2, E^2) . (This relation can be seen immediately to be an equivalence relation.)

Let us recall the definition of the category of Bratteli diagrams \mathbf{BD}_1 , as introduced in [1], in a way suited to Definition 2.1. As we shall see, the notion of isomorphism of Herman, Putnam, and Skau coincides with isomorphism in the category \mathbf{BD}_1 with premorphisms (Proposition 2.7), and their notion of equivalence coincides with isomorphism in the category \mathbf{BD}_1 with morphisms (Theorem 2.11).

Definition 2.4 (cf. [1], Definition 2.3). Let $B = (V, E)$ and $C = (W, S)$ be in \mathbf{BD}_1 . A *premorphism* $f : B \rightarrow C$ is an ordered pair $(F, (f_n)_{n=0}^\infty)$ where $(f_n)_{n=0}^\infty$ is a cofinal sequence of positive integers with $f_0 = 0 \leq f_1 \leq f_2 \leq \dots$ and F consists of a disjoint union $F_0 \cup F_1 \cup F_2 \cup \dots$ together with a pair of range and source maps $r : F \rightarrow W$, $s : F \rightarrow V$ such that:

- (1) each F_n is a non-empty finite set, $s(F_n) \subseteq V_n$, $r(F_n) \subseteq W_{f_n}$, F_0 is a singleton, $s^{-1}\{v\}$ is non-empty for all v in V , and $r^{-1}\{w\}$ is non-empty for all w in W ;
- (2) the diagram of $f : B \rightarrow C$ commutes:

$$\begin{array}{ccccccc} V_0 & \xrightarrow{E_1} & V_1 & \xrightarrow{E_2} & V_2 & \xrightarrow{E_3} & \dots \\ F_0 \downarrow & & F_1 \downarrow & \swarrow F_2 & & & \\ W_0 & \xrightarrow{S_1} & W_1 & \xrightarrow{S_2} & W_2 & \xrightarrow{S_3} & \dots \end{array}$$

By commutativity of the diagram, we mean that for each $n \geq 0$, each $v \in V_n$, and each $w \in W_{f_{n+1}}$, the number of paths from v to w passing through W_{f_n} equals that of ones passing through V_{n+1} ; we use the notation $E_n \circ F_{n+1} \cong F_n \circ S_{f_n, f_{n+1}}$ to refer to this fact.

Remark. With the notation of Definition 2.4, the diagram of $f : B \rightarrow C$ commutes if, and only if, for any positive integer n we have $M(F_{n+1})M(E_n) = M(S_{f_n, f_{n+1}})M(F_n)$, i.e., the square

$$\begin{array}{ccc} V_n & \xrightarrow{M(E_n)} & V_{n+1} \\ M(F_n) \downarrow & & \downarrow M(F_{n+1}) \\ W_{f_n} & \xrightarrow{M(S_{f_n, f_{n+1}})} & W_{f_{n+1}} \end{array}$$

commutes. (This implies the general property of commutativity, namely, that any two paths of maps between the same pair of points in the diagram agree, i.e., have the same product.)

In order to obtain a category of Bratteli diagrams with premorphisms, as defined here (this difficulty did not arise with the definition of [1], i.e., Definition 2.2 above), we need to divide out by a fairly trivial equivalence relation on premorphisms. (In Definition 2.8 we will define a much stronger notion of equivalence which leads to the notion of morphism of Bratteli diagram.)

Definition 2.5. Let $B, C \in \mathbf{BD}_1$ and $f, f' : B \rightarrow C$ be a pair of premorphisms where $f = (F, (f_n)_{n=0}^\infty)$ and $f' = (F', (f'_n)_{n=0}^\infty)$. Let us say that f is *isomorphic* to f' , and write $f \cong f'$, if $f_n = f'_n$, $n \geq 0$, and there is a bijective map from F to F' preserving the range and source maps. The relation \cong is an equivalence relation on the class of all premorphisms from B to C . Let us denote the equivalence class of f by \bar{f} . Let B, C , and D be objects in \mathbf{BD}_1

and let $f : B \rightarrow C$ and $g : C \rightarrow D$ be premorphisms, $f = (F, (f_n)_{n=0}^\infty)$ and $g = (G, (g_n)_{n=0}^\infty)$ where $F = \bigcup_{n=0}^\infty F_n$ and $G = \bigcup_{n=0}^\infty G_n$ (disjoint unions). The composition of f and g is defined as $gf = (H, (h_n)_{n=0}^\infty)$, where $h_n = g_{f_n}$, $H = \bigcup_{n=0}^\infty H_n$, and $H_n = F_n \circ G_{f_n}$, $n \geq 0$ (i.e., the set of all paths from $s(F_n)$ to $r(G_{f_n})$; see the remarks before Definition 2.3). Also, set $\overline{gf} = \overline{g}\overline{f}$.

Proposition 2.6. \mathbf{BD}_1 , with premorphisms modulo the relation (isomorphism) of Definition 2.5, is a category. (Let us refer to this as the category of Bratteli diagrams with premorphisms.)

Proof. The proof is essentially the same as that of [1, Proposition 2.4]. Note that, although the relation $(fg)h = f(gh)$ does not strictly speaking hold for premorphisms, we do have $(\overline{f}\overline{g})\overline{h} = \overline{f}(\overline{g}\overline{h})$. \square

Proposition 2.7. Two Bratteli diagrams are isomorphic in the category \mathbf{BD}_1 with premorphisms (of Proposition 2.6) if, and only if, they are isomorphic in the sense of Herman, Putnam, and Skau.

Proof. Suppose that $B = (V, E)$ and $C = (W, S)$ are Bratteli diagrams which are isomorphic in the category \mathbf{BD}_1 with premorphisms. Thus there are premorphisms $f : B \rightarrow C$ and $g : C \rightarrow B$ such that $fg \cong \text{id}_C$ and $gf \cong \text{id}_B$. Write $f = (F, (f_n)_{n=0}^\infty)$ and $g = (G, (g_n)_{n=0}^\infty)$. We claim that $f_n = g_n = n$, for each $n \geq 0$. Let us use induction on n . For $n = 0$, by Definition 2.4, $f_0 = g_0 = 0$. Suppose that $n \geq 1$ and for each $1 \leq k \leq n-1$, $f_k = g_k = k$. Since $(f_n)_{n=0}^\infty$ and $(g_n)_{n=0}^\infty$ are increasing sequences, $f_n, g_n \geq n-1$. If $f_n = n-1$ then $g_{f_n} = g_{n-1} = n-1$, which is not the case. Hence $f_n \geq n$ and by symmetry $g_n \geq n$. If $f_n > n$ then $g_n > n$. Then $n = f_{g_n} \geq f_n > n$ which is a contradiction. Therefore, $f_n = g_n = n$ and the claim is proved.

Fix $n \geq 1$. We have $F_n \circ G_n \cong \text{id}_{W_n}$ and $G_n \circ F_n \cong \text{id}_{V_n}$. Thus $M(F_n)M(G_n) = I_{|W_n|}$ and $M(G_n)M(F_n) = I_{|V_n|}$ where $|W_n|$ denotes the cardinality of W_n and similarly for V_n . Since the entries of the matrices $M(G_n)$ and $M(F_n)$ are positive (i.e., non-negative) integers, we have $|V_n| = |W_n|$ and the entries of $M(G_n)$ and $M(F_n)$ are 0 or 1 with exactly one 1 in each row and each column. Using this and commutativity of the diagram of f as specified in Definition 2.4, it is not hard to see that there exist bijections between V and W and between E and S , preserving the gradings and intertwining the respective source and range maps.

The proof of the backward implication is immediate. \square

Definition 2.8. Let B, C be Bratteli diagrams and $f, g : B \rightarrow C$ be premorphisms with $B = (V, E)$, $C = (W, S)$, $f = (F, (f_n)_{n=0}^\infty)$, and $g = (G, (g_n)_{n=0}^\infty)$. Let us say that f is *equivalent* to g , and write $f \sim g$, if there are sequences $(n_k)_{k=1}^\infty$ and $(m_k)_{k=1}^\infty$ of positive integers such that $n_k < m_k < n_{k+1}$ and $f_{n_k} < g_{m_k} < f_{n_{k+1}}$ for each $k \geq 1$, and the diagram

$$\begin{array}{ccccccc}
V_{n_1} & \longrightarrow & V_{m_1} & \longrightarrow & V_{n_2} & \longrightarrow & V_{m_2} \longrightarrow \cdots \\
F_{n_1} \downarrow & & G_{m_1} \downarrow & & F_{n_2} \downarrow & & G_{m_2} \downarrow \\
W_{f_{n_1}} & \longrightarrow & W_{g_{m_1}} & \longrightarrow & W_{f_{n_2}} & \longrightarrow & W_{g_{m_2}} \longrightarrow \cdots
\end{array}$$

commutes, i.e., each minimal square commutes: for each $k \geq 1$,

$$E_{n_k, m_k} \circ G_{m_k} \cong F_{n_k} \circ S_{f_{n_k}, g_{m_k}},$$

$$E_{m_k, n_{k+1}} \circ F_{n_{k+1}} \cong G_{m_k} \circ S_{g_{m_k}, f_{n_{k+1}}}.$$

At the end of this section we will give two other definitions for equivalence of premorphisms (Definitions 2.12 and 2.13). These may be more natural in some sense (since they do not use subsequences), but we shall show that all three equivalence relations are the same (Proposition 2.14).

Lemma 2.9. *Let $B, C \in \mathbf{BD}_1$. Then \sim is an equivalence relation on the class of premorphisms from B to C .*

Proof. In view of the remark preceding Definition 2.5, the proof is essentially the same as that of [1, Lemma 2.6]. \square

Let us call an equivalence class of premorphisms between Bratteli diagrams B and C , with respect to the relation \sim , a *morphism* from B to C . Let us denote the equivalence class of a premorphism $f : B \rightarrow C$ by $[f] : B \rightarrow C$, or if there is no confusion, just by f .

The composition of morphisms $[f] : B \rightarrow C$ and $[g] : C \rightarrow D$ should be defined as $[gf] : B \rightarrow D$ where gf is the composition of premorphisms. This composition is well defined, as is affirmed in the following theorem.

Theorem 2.10. *The class \mathbf{BD}_1 , with morphisms as defined above, is a category.*

Proof. The proof is similar to that of [1, Theorem 2.7] (use the remark preceding Definition 2.5). \square

Let us refer to \mathbf{BD}_1 , with morphisms as defined above, as the *category of Bratteli diagrams*.

Theorem 2.11. *Two Bratteli diagrams are isomorphic in the category \mathbf{BD}_1 (Theorem 2.10) if, and only if, they are equivalent in the sense of Herman, Putnam, and Skau.*

Proof. The statement follows from Theorem 2.15 and [1, Theorem 4.5]. \square

Here are two alternative formulations of the definition for equivalence of premorphisms (Definition 2.8). We shall use the first one in a number of places later.

Definition 2.12. Let $f, g : B \rightarrow C$ be premorphisms in \mathbf{BD}_1 such that $B = (V, E)$, $C = (W, S)$, $f = (F, (f_n)_{n=0}^\infty)$, and $g = (G, (g_n)_{n=0}^\infty)$. Let us say that f is *equivalent* to g , in the second sense, if for each $n \geq 0$ there is an $m \geq f_n, g_n$ such that $F_n \circ S_{f_n, m} \cong G_n \circ S_{g_n, m}$.

Definition 2.13. Let $f, g : B \rightarrow C$ be premorphisms in \mathbf{BD}_1 such that $B = (V, E)$, $C = (W, S)$, $f = (F, (f_n)_{n=0}^\infty)$, and $g = (G, (g_n)_{n=0}^\infty)$. Let us say that f is *equivalent* to g , in the third sense, if for each $n \geq 0$ and for each $k \geq n$, there is an $m \geq f_n, g_k$ such that $F_n \circ S_{f_n, m} \cong E_{n, k} \circ G_k \circ S_{g_k, m}$.

Let us show that these two definitions are equivalent to Definition 2.8.

Proposition 2.14. *Definitions 2.8, 2.12, and 2.13 are equivalent.*

Proof. The proof is essentially the same as that of [1, Proposition 2.11]. \square

Let us now show that the categories \mathbf{BD}^1 and \mathbf{BD}_1 are equivalent. Define the map $\mathcal{F} : \mathbf{BD}^1 \rightarrow \mathbf{BD}_1$ as follows. Let $B = (V, E)$ be in \mathbf{BD}^1 (see Definition 2.2 and the following remarks). Set $\mathcal{F}(B) = (V', E')$ where $V' = \bigcup_{n=0}^\infty V'_n$, $E' = \bigcup_{n=1}^\infty E'_n$, $V'_0 = \{(0, 1, 1)\}$, and for $n \geq 1$, $V'_n = \{(n, i, a_i) \mid 1 \leq i \leq k_n\}$ where a_1, a_2, \dots, a_{k_n} are entries of V_n . Moreover, E'_1 consists of all $(e, f, 1), (e, f, 2), \dots, (e, f, m)$ where e is the element of V'_0 , $f \in V'_1$, and m is the number of paths from e to f determined by V_1 , i.e., m is the third component of f . For $n \geq 2$, E'_n consists of all $(e, f, 1), (e, f, 2), \dots, (e, f, m)$ where $e \in V'_{n-1}$, $f \in V'_n$, and m is the number of paths from e to f determined by E_{n-1} , i.e., m is the j th entry of E_{n-1} where i and j are the second components of e and f , respectively.

Let us define the map $\mathcal{F} : \mathbf{BD}^1 \rightarrow \mathbf{BD}_1$ on premorphisms as follows. Let $f = ((F_n)_{n=1}^\infty, (f_n)_{n=1}^\infty)$ be a premorphism from $B = (V, E)$ to $C = (W, S)$ in \mathbf{BD}^1 as in [1, Definition 2.3]. Set $\mathcal{F}(f) = ((F'_n)_{n=0}^\infty, (f'_n)_{n=0}^\infty)$ where $f_0 = 0$, $F'_0 = \{(0, 0, 1)\}$, and for each $n \geq 1$, F'_n consists of all $(e, e', 1), (e, e', 2), \dots, (e, e', m)$ where $e \in V'_n$, $e' \in W_{f_n}$, and m is the number of paths from e to e' determined by F_n , i.e., m is the j th entry of F_n where i and j are the second components of e and e' , respectively. It is easy to see that $\mathcal{F}(f)$ is a premorphism in \mathbf{BD}_1 as in Definition 2.4. Also, \mathcal{F} maps equivalent premorphisms of \mathbf{BD}^1 to equivalent premorphisms of \mathbf{BD}_1 . Hence we can define \mathcal{F} on morphisms, that is, set $\mathcal{F}([f]) = [\mathcal{F}(f)]$.

Theorem 2.15. *The map \mathcal{F} from the category \mathbf{BD}^1 with morphisms to the category \mathbf{BD}_1 with morphisms, as defined above, is an equivalence of categories.*

Proof. First note that \mathcal{F} is well defined, that is, \mathcal{F} preserves the identity and composition. Also it is easily verified that \mathcal{F} is full, faithful, and essentially surjective. Therefore, by Theorem 4.13, $\mathcal{F} : \mathbf{BD}^1 \rightarrow \mathbf{BD}_1$ is an equivalence of categories. \square

Remark. Define the map $\overline{\mathcal{F}} : \mathbf{BD}^1 \rightarrow \mathbf{BD}_1$ from the category \mathbf{BD}^1 with premorphisms (see the remarks following Definition 2.2) to the category \mathbf{BD}_1

with equivalence classes of premorphisms (see Definition 2.5 and Proposition 2.6) as follows. For each $B \in \mathbf{BD}^1$ set $\overline{\mathcal{F}}(B) = \mathcal{F}(B)$ and for each premorphism f in \mathbf{BD}^1 set $\overline{\mathcal{F}}(f) = \overline{\mathcal{F}(f)}$. Then $\overline{\mathcal{F}}$ is a functor which is full, faithful, and essentially surjective. Therefore, it is an equivalence of categories (by Theorem 4.13).

3. THE CATEGORY OF ORDERED BRATTELI DIAGRAMS

In this section we propose a notion of morphism for the class of ordered Bratteli diagrams, to construct a category of ordered Bratteli diagrams, which we shall denote by **OBD**. This construction is very similar to the construction of the category of Bratteli diagrams \mathbf{BD}_1 given in the previous section. In particular, first we need the notion of premorphism.

We shall show that isomorphism in this category coincides with equivalence of ordered Bratteli diagrams as defined by Herman, Putnam, and Skau in [12] (see Theorem 3.6 below).

Definition 3.1 ([12], Definition 2.3). An *ordered Bratteli diagram* is a Bratteli diagram (V, E) together with an order relation \geq on E such that e and e' are comparable if and only if $r(e) = r(e')$. In other words, we have a linear order on each set $r^{-1}\{v\}$, $v \in V \setminus V_0$.

The order of an ordered Bratteli diagram induces an order on its contractions. In fact, if (V, E, \geq) is an ordered Bratteli diagram and k, l are integers with $0 \leq k < l$, then the set $E_{k+1} \circ E_k \circ \cdots \circ E_l$ of all paths from V_k to V_l may be given an induced (lexicographic) order as follows:

$$(e_{k+1}, e_{k+2}, \dots, e_l) > (f_{k+1}, f_{k+2}, \dots, f_l)$$

if for some i with $k+1 \leq i \leq l$, $e_j = f_j$ for $i < j \leq l$ and $e_i > f_i$. If (V, E, \geq) is an ordered Bratteli diagram and (V', E') is a contraction of (V, E) as defined above, then with the induced order \geq' , (V', E', \geq') is again an ordered Bratteli diagram which is called a *contraction* (*telescoping*) of (V, E, \geq) [4, 12].

Herman, Putnam, and Skau consider a notion of isomorphism of ordered Bratteli diagrams (similar to their notion of isomorphism of Bratteli diagrams). Also, they denote by \approx the equivalence relation on ordered Bratteli diagrams generated by isomorphism and by telescoping [12]. One can see that $B^1 \approx B^2$, where $B^1 = (V^1, E^1, \geq^1)$ and $B^2 = (V^2, E^2, \geq^2)$, if and only if there exists an ordered Bratteli diagram $B = (V, E, \geq)$ such that telescoping B to odd levels, $0 < 1 < 3 < \cdots$, yields a telescoping of B^1 , and telescoping B to even levels, $0 < 2 < 4 < \cdots$, yields a telescoping of B^2 [4]. This is analogous to the situation for the equivalence relation \sim on Bratteli diagrams as we discussed in Section 2.

Let us define the category of ordered Bratteli diagrams. Just as in the construction of the category of Bratteli diagrams in Section 2, we need a notion of premorphism before considering the final notion of morphism (to

be called ordered morphism in the sequel) in this category. Denote by **OBD** the class of all ordered Bratteli diagrams.

Definition 3.2 (cf. Definition 2.4). Let $B = (V, E, \geq)$ and $C = (W, S, \geq)$ be ordered Bratteli diagrams. By an (ordered) *premorphisms* $f : B \rightarrow C$ we mean a triple $(F, (f_n)_{n=0}^\infty, \geq)$ where $(F, (f_n)_{n=0}^\infty)$ is a premorphism according to Definition 2.4, and \geq is a partial order on F such that:

- (1) $e, e' \in F$ are comparable if and only if $r(e) = r(e')$, and \geq is a linear order on $r^{-1}\{v\}$, $v \in W$;
- (2) the diagram of $f : B \rightarrow C$ commutes:

$$\begin{array}{ccccccc} V_0 & \xrightarrow{E_1} & V_1 & \xrightarrow{E_2} & V_2 & \xrightarrow{E_3} & \cdots \\ F_0 \downarrow & & F_1 \downarrow & \swarrow F_2 & & & \\ W_0 & \xrightarrow{S_1} & W_1 & \xrightarrow{S_2} & W_2 & \xrightarrow{S_3} & \cdots \end{array}$$

Commutativity of the diagram of course amounts to saying that for each $n \geq 0$, $E_n \circ F_{n+1} \cong F_n \circ S_{f_n, f_{n+1}}$, which means that there is a (necessarily unique) bijective map from $E_n \circ F_{n+1}$ to $F_n \circ S_{f_n, f_{n+1}}$ preserving the order and intertwining the respective source and range maps.

Similar to Definition 2.5, an isomorphism relation on the class of ordered premorphisms is defined (considering order). The proofs of the following propositions are similar to their analogues in Section 2 (Propositions 2.6 and 2.7).

Proposition 3.3. *The class **OBD**, with (ordered) premorphisms modulo the relation (isomorphism) stated above, is a category. (Let us refer to this as the category of ordered Bratteli diagrams with premorphisms.)*

Proposition 3.4. *Two ordered Bratteli diagrams are isomorphic in the category **OBD** with ordered premorphisms if, and only if, they are isomorphic in the sense of Herman, Putman, and Skau.*

We may define an equivalence relation on ordered premorphisms in a way similar to Definition 2.8, considering order. Let us call the equivalence classes *ordered morphisms* (or if there is no confusion, just *morphisms*) in **OBD**. Note that one can also formulate the alternative Definitions 2.12 and 2.13 for ordered premorphisms (considering order), and again the analogue of Proposition 2.14 holds, for ordered premorphisms.

The proofs of the following theorems are similar to their analogues in Section 2 (Theorems 2.10 and 2.11).

Theorem 3.5. *The class **OBD** with (ordered) morphisms as defined above is a category.*

Theorem 3.6. *Two ordered Bratteli diagrams are isomorphic in the category **OBD** if, and only if, they are equivalent in the sense of Herman, Putnam, and Skau.*

Let us refer to the category **OBD** with (ordered) morphisms as defined above as the *category of ordered Bratteli diagrams*.

4. THE FUNCTOR \mathcal{P} FROM **SDS** TO **OBD**

In this section, first we define the category of scaled essentially minimal totally disconnected dynamical systems **SDS**. Then we construct a functor \mathcal{P} from **SDS** to the category of ordered Bratteli diagrams **OBD**. As we shall show, the essential range of this functor is the full subcategory of **OBD** consisting of essentially simple ordered Bratteli diagrams **OBD**_{es}. We will show that $\mathcal{P} : \mathbf{SDS} \rightarrow \mathbf{OBD}_{\text{es}}$ is an equivalence of categories (Theorem 4.14 below).

In this way, we obtain a functorial formulation and a generalization of one of the main results of Herman, Putnam, and Skau [12, Theorem 4.7]. In fact, they considered only the isomorphism classes of these categories, but our approach takes into account the morphisms of these categories.

Here we are mainly interested in the minimal systems, but almost all the results hold in a more general setting, i.e., essentially minimal totally disconnected systems. However, the minimal case will be discussed specifically at the end of Section 5.

Definition 4.1 (cf. [12], Definition 1.2). Let X be a compact metrizable space. Let φ be a homeomorphism of X and $y \in X$. The triple (X, φ, y) is called an *essentially minimal dynamical system* if the dynamical system (X, φ) has a unique minimal (non-empty, closed, invariant) set Y and $y \in Y$.

Remark. Note that if X is totally disconnected and has no isolated points, then X is homeomorphic to the Cantor set. Moreover, there are essentially minimal totally disconnected dynamical systems which are not minimal. For example, the one-point compactification of a locally compact Cantor minimal system is essentially minimal but not minimal ([14]).

Definition 4.2. Let us define the category of essentially minimal totally disconnected dynamical systems **DS** as follows. The objects of this category are the essentially minimal totally disconnected dynamical systems. Let (X, φ, y) and (Y, ψ, z) be in **DS**. By a morphism $\alpha : (X, \varphi, y) \rightarrow (Y, \psi, z)$ in **DS** we mean a homomorphism from the dynamical system (X, φ) to (Y, ψ) (i.e., a continuous map with $\alpha \circ \varphi = \psi \circ \alpha$) such that $\varphi(y) = z$.

For the rest of the paper, we shall use the abbreviation t.d. for totally disconnected.

Remark. Note that in the above definition, α maps the unique minimal set of (X, φ) to that of (Y, ψ) . Also, isomorphism in the category **DS** coincides with the notion of pointed topological conjugacy introduced in [12].

Let us recall the notion of a Kakutani-Rokhlin partition.

Definition 4.3 ([12]). Let (X, φ, y) be an essentially minimal t.d. dynamical system. A *Kakutani-Rokhlin partition* for (X, φ, y) is a partition P of X where

$$P = \{Z(k, j) \mid k = 1, \dots, K, j = 1, \dots, J(k)\},$$

in which K and $J(1), \dots, J(K)$ are non-zero positive integers and the $Z(k, j)$ are non-empty clopen subsets of X with the following properties:

- (1) $\varphi(Z(k, j)) = Z(k, j+1)$ for all $1 \leq k \leq K$, and $1 \leq j < J(k)$;
- (2) setting $Z = \bigcup_k Z(k, J(k))$, we have $y \in Z$ and $\varphi(Z) = \bigcup_k Z(k, 1)$.

For each $1 \leq k \leq K$, the set $\{Z(k, j) \mid j = 1, \dots, J(k)\}$ is called the k th *tower* of P with *height* $J(k)$. The sets Z and $\varphi(Z)$ are called the *top* and *base* of P , respectively.

The idea of the following definition was used implicitly in [12].

Definition 4.4. Let (X, φ, y) be an essentially minimal t.d. dynamical system. A *system of Kakutani-Rokhlin partitions* for (X, φ, y) is a sequence $\{P_n\}_0^\infty$ of Kakutani-Rokhlin partitions for X such that $P_0 = \{X\}$ and:

- (1) if Z_n denotes the top of P_n for each $n \geq 1$, the sequence $\{Z_n\}_{n=1}^\infty$ is a decreasing sequence of clopen sets with intersection $\{y\}$;
- (2) for all n , $P_n \leq P_{n+1}$, i.e., P_{n+1} is a refinement of P_n ;
- (3) $\bigcup_{n=0}^\infty P_n$ is a base for the topology of X .

Remark. For each essentially minimal t.d. dynamical system (X, φ, y) , partitions as in Definition 4.4 exist in abundance. In fact, let $\{Z_n\}_{n=1}^\infty$ be a decreasing sequence of clopen sets with intersection $\{y\}$. Then, by [12, Theorem 4.2], there is a (not necessarily unique) system of Kakutani-Rokhlin partitions for (X, φ, y) with Z_n the top of the n th partition.

Definition 4.5. By a *scaled essentially minimal t.d. dynamical system* we mean a quadruple $(X, \varphi, y, \mathcal{R})$ where (X, φ, y) is an essentially minimal t.d. dynamical system and \mathcal{R} is a system of Kakutani-Rokhlin partitions for (X, φ, y) . The *category of scaled essentially minimal t.d. dynamical systems* **SDS** is the category whose objects is the class of all essentially minimal t.d. dynamical systems and its morphisms are as follows. Let $(X, \varphi, y, \mathcal{R})$ and $(Y, \psi, z, \mathcal{S})$ be in **SDS**. By a morphism $\alpha : (X, \varphi, y, \mathcal{R}) \rightarrow (Y, \psi, z, \mathcal{S})$ we mean a homomorphism from the dynamical system (X, φ) to (Y, ψ) (i.e., a continuous map with $\alpha \circ \varphi = \psi \circ \alpha$) such that $\varphi(y) = z$.

We shall need the following notation in a number of places.

Notation. Let (X, φ, y) be an essentially minimal t.d. dynamical system and P and Q be a pair of Kakutani-Rokhlin partitions for it such that $P \leq Q$ (i.e., Q is a refinement of P) and the top of P contains the top of Q . Considering the elements of P and Q as vertices, we denote by $E(P, Q)$ the (ordered) edge set from P to Q defined as follows. We have an edge in $E(P, Q)$ each time a tower of Q passes a tower of P ; specifically, $E(P, Q)$ contains all elements of the form (S, T, k) where $S = \{Z_1, \dots, Z_n\}$ and

$T = \{Y_1, \dots, Y_m\}$ are towers of P and Q , respectively, and k is a non-negative integer such that $Y_{k+j} \subseteq Z_j$ for all $1 \leq j \leq n$ (cf. [12, Section 4]). Note that $(S, T, k) \in E(P, Q)$ if and only if $Y_{k+j} \subseteq Z_j$ for some $1 \leq j \leq n$. Write $(S, T, k) \leq (S', T', k')$ if $T = T'$ and $k \leq k'$. This is a partial order on $E(P, Q)$, which is a total order on the set of edges leading to a common vertex.

We shall also need the following lemma. This is a topological version of [1, Lemma 3.4] (see Definition 3.2 for the notation \cong).

Lemma 4.6. *Let (X, φ, y) be an essentially minimal t.d. dynamical system and let P_1, P_2 , and P_3 be Kakutani-Rokhlin partitions for it such that $P_1 \leq P_2 \leq P_3$ and the top of P_i contains the top of P_{i+1} , for $i = 1, 2$. Then we have $E(P_1, P_3) \cong E(P_1, P_2) \circ E(P_2, P_3)$, i.e., the following diagram commutes:*

$$\begin{array}{ccc} P_1 & \xrightarrow{E(P_1, P_2)} & P_2 \\ & \searrow E(P_1, P_3) & \downarrow E(P_2, P_3) \\ & & P_3 \end{array}$$

Proof. The lemma follows from the following fact. Let P and Q be a pair of Kakutani-Rokhlin partitions for (X, φ, y) such that $P \leq Q$ and the top of P contains the top of Q . Let $T = \{Y_1, \dots, Y_n\}$ be a tower of Q . Then there are unique integers $J_0 = 0 < J_1 < J_2 < \dots < J_m = n$ and unique towers S_1, S_2, \dots, S_m of P with the following properties. The height of S_i equals $h_i = J_i - J_{i-1}$, $1 \leq i \leq m$, and if we set $S_i = \{Z(i, 1), Z(i, 2), \dots, Z(i, h_i)\}$ then $Y_{J_{i-1}+j} \subseteq Z(i, j)$ for all $1 \leq j \leq h_i$. Thus, $(S_i, T, J_{i-1}) \in E(P, Q)$ for all $1 \leq i \leq m$, and $(S_1, T, J_0) < (S_2, T, J_1) < \dots < (S_m, T, J_{m-1})$. \square

Definition 4.7. Define the contravariant functor $\mathcal{P} : \mathbf{SDS} \rightarrow \mathbf{OBD}$ as follows. Let $(X, \varphi, y, \mathcal{R})$ be in \mathbf{SDS} . Consider the ordered Bratteli diagram $\mathcal{P}(X, \varphi, y, \mathcal{R}) = (V, E, \geq)$ constructed in [12, Section 4] for $(X, \varphi, y, \mathcal{R})$. To recall, let \mathcal{R} be as in Definition 4.4 and set $V_n = \{(n, T) \mid T \text{ is a tower of } P_n\}$, $n \geq 0$, and $V = \bigcup_{n=0}^{\infty} V_n$. Set $E_n = \{(n, S, T, k) \mid (S, T, k) \in E(P_{n-1}, P_n)\}$, $n \geq 1$, and $E = \bigcup_{n=1}^{\infty} E_n$. The partial order on E is defined as the union of partial orderings on the E_n as described just before Lemma 4.6.

Now let $(X, \varphi, y, \mathcal{R})$ and $(Y, \psi, z, \mathcal{S})$ be in \mathbf{SDS} where $\mathcal{R} = \{P_n\}_{n=0}^{\infty}$ and $\mathcal{S} = \{Q_n\}_{n=0}^{\infty}$. Let $\alpha : (X, \varphi, y, \mathcal{R}) \rightarrow (Y, \psi, z, \mathcal{S})$ be a morphism in \mathbf{SDS} , i.e., a continuous map from X to Y with $\alpha(y) = z$ and $\alpha \circ \varphi = \psi \circ \alpha$. Define the ordered premorphism $f = (F, (f_n)_{n=0}^{\infty}, \geq)$ from $\mathcal{P}(Y, \psi, z, \mathcal{S}) = (W, S, \geq)$ to $\mathcal{P}(X, \varphi, y, \mathcal{R}) = (V, E, \geq)$ as follows.

Set $f_0 = 0$ and $F_0 = \{0\}$. Suppose that we have chosen f_0, f_1, \dots, f_{n-1} and F_0, F_1, \dots, F_{n-1} and define f_n and F_n as follows. Since Q_n is a Kakutani-Rokhlin partition for (Y, ψ, z) , the set $\alpha^{-1}(Q_n)$ of inverse images of elements of Q_n is a Kakutani-Rokhlin partition for (X, φ, y) . By the properties (1) and (3) of Definition 4.4, there is an integer f_n with $f_n \geq f_{n-1}$ such that

$\alpha^{-1}(Q_n) \leq P_{f_n}$ and the top of $\alpha^{-1}(Q_n)$ contains the top of P_{f_n} . Set

$$F_n = \{(n, S, T, k) \mid (\alpha^{-1}(S), T, k) \in E(\alpha^{-1}(Q_n), P_{f_n})\}.$$

There is a one-to-one correspondence between the elements of F_n and those of $E(\alpha^{-1}(Q_n), P_{f_n})$. Define the order on F_n to be the induced order from $E(\alpha^{-1}(Q_n), P_{f_n})$. In this way F_n becomes an edge set from W_n to V_{f_n} .

Continuing this procedure, we obtain a cofinal sequence of integers $(f_n)_{n=0}^\infty$ with $f_0 = 0 \leq f_1 \leq f_2 \leq \dots$ and an edge set $F = \bigcup_{n=0}^\infty F_n$ such that each F_n is an edge set from W_n to V_{f_n} . The source and range maps are defined in the natural way, i.e., $s(n, S, T, k) = (n, S)$ and $r(n, S, T, k) = (f_n, T)$. The partial order \leq on F is the union of the partial orders on the F_n . Now set $f = (F, (f_n)_{n=0}^\infty, \geq)$. Applying Lemma 4.6, we see that $f : (W, S) \rightarrow (V, E)$ is an (ordered) premorphism. Set $\mathcal{P}(\alpha) = [f]$ the equivalence class of f .

Theorem 4.8. *The map $\mathcal{P} : \mathbf{SDS} \rightarrow \mathbf{OBD}$, as defined above, is a contravariant functor.*

Proof. First note that \mathcal{P} is single-valued, i.e., with the notation as above, $\mathcal{P}(\alpha)$ is independent of the choice of the sequence $(f_n)_{n=0}^\infty$. To see this, let $(g_n)_{n=0}^\infty$ be another cofinal sequence with $g_0 = 0 \leq g_1 \leq g_2 \leq \dots$ which gives the premorphism $g = (G, (g_n)_{n=0}^\infty)$ according to Definition 4.7. Fix $n \geq 1$ and assume, without loss of generality, that $f_n \leq g_n$. Then $\alpha^{-1}(Q_n) \leq P_{f_n} \leq P_{g_n}$. By Lemma 4.6, the diagram

$$\begin{array}{ccc} W_n & & \\ \downarrow F_n \quad \searrow G_n & & \\ V_{f_n} & \xrightarrow{E_{f_n, g_n}} & V_{g_n} \end{array}$$

commutes, which means that $F_n \cong G_n \circ E_{f_n, g_n}$. By the analogous version of Proposition 2.14 for ordered premorphisms, we have $f \sim g$.

The proof of the fact that \mathcal{P} preserves composition (as a contravariant functor) is similar to the last part of the proof of [1, Proposition 3.8], but much easier (use Lemma 4.6). \square

The following theorem is a constructive result. In fact, it is shown that each premorphism between the Bratteli diagrams of two essentially minimal t.d. dynamical systems can be lifted back to a homomorphism between them. This was shown for the special case that the Vershik transformation of the Bratteli diagram of an essentially minimal t.d. dynamical system is isomorphic to that system [12, Theorem 4.6].

Theorem 4.9. *The functor $\mathcal{P} : \mathbf{SDS} \rightarrow \mathbf{OBD}$ is a full functor.*

Proof. The idea of the proof is to reverse the procedure of Definition 4.7. Let $\mathcal{X}_1 = (X, \varphi, y, \mathcal{R})$ and $\mathcal{X}_2 = (Y, \psi, z, \mathcal{S})$ be in \mathbf{SDS} and write $\mathcal{P}(\mathcal{X}_1) = (V, E)$ and $\mathcal{P}(\mathcal{X}_2) = (W, S)$. Let $f : (W, S) \rightarrow (V, E)$ be an ordered premorphism. We must show that there is a morphism $\alpha : \mathcal{X}_1 \rightarrow \mathcal{X}_2$ with $\mathcal{P}(\alpha) = [f]$.

Write $f = (F, (f_n)_{n=0}^\infty, \geq)$, $\mathcal{R} = \{P_n\}_{n=0}^\infty$, and $\mathcal{S} = \{Q_n\}_{n=0}^\infty$. Let $F = \bigcup_{n=0}^\infty F_n$ denote the decomposition of F according to Definition 3.2. We have the following fact. For each $n \geq 0$, F_n fills the towers of P_{f_n} with the towers of Q_n ; specifically, let T be a tower of P_{f_n} . Let e_1, e_2, \dots, e_k denote the edges in F_n with range (f_n, T) and $e_1 < e_2 < \dots < e_k$. Denote by S_i the tower of Q_n such that the source of e_i is (n, S_i) , $1 \leq i \leq k$. Then the height of T equals the sum of the heights of S_1, S_2, \dots, S_k . (This follows from the fact that f is a premorphism by induction on n .)

Choose x in X . For each $n \geq 0$ there is $A_n \in P_{f_n}$ such that $x \in A_n$. We have $A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$. Since X is Hausdorff and $\bigcup_{n=0}^\infty P_n$ is a basis for X , we have $\bigcap_{n=0}^\infty A_n = \{x\}$. Fix $n \geq 0$. With T_n the tower of P_{f_n} containing A_n , by the preceding paragraph, there is a unique tower S_n in Q_n and a unique element B_n in S_n which corresponds to A_n when F_n fills T_n by the towers of Q_n . (We will construct α in such a way that $\alpha(A_n) \subseteq B_n$.) By Definition 3.2, for each $n \geq 1$, we have $E_{n-1} \circ F_n \cong F_{n-1} \circ S_{f_{n-1}, f_n}$. Using this, one can see that $B_n \subseteq B_{n-1}$. The set $\bigcap_{n=0}^\infty B_n$ is a singleton; let $\alpha(x)$ denote the element of this set. Thus we have a map $\alpha : X \rightarrow Y$.

Our construction yields that $\alpha(A_n) \subseteq B_n$, $n \geq 0$. With this it is not hard to see that α is continuous. Let us show that $\alpha(y) = z$. Since y is in the top of each P_{f_n} , $\alpha(y)$ is in the top of each Q_n . Now by the property (1) in Definition 4.4 we have $\alpha(y) = z$.

Finally let us show that $\alpha \circ \varphi = \psi \circ \alpha$. Let $x \in X \setminus \{y\}$. Choose A_n 's and B_n 's as above. By the property (1) of Definition 4.4, there is $n_0 \geq 0$ such that A_n is not contained in the top of P_{f_n} , for each $n \geq n_0$. Hence B_n is not contained in the top of Q_n , $n \geq n_0$. This shows that $\varphi(A_n)$ and $\psi(B_n)$ are elements of P_{f_n} and Q_n which are located above A_n and B_n , respectively, and hence, by the definition of α , $\alpha(\varphi(A_n)) \subseteq \psi(B_n)$. Thus $\alpha(\varphi(x)) \in \psi(B_n)$, $n \geq n_0$. On the other hand, $\psi(\alpha(x)) \in \psi(B_n)$. Hence, $\{\alpha(\varphi(x))\} = \bigcap_{n=n_0}^\infty \psi(B_n) = \{\psi(\alpha(x))\}$, i.e., $\alpha(\varphi(x)) = \psi(\alpha(x))$. That $\alpha(\varphi(y)) = \psi(\alpha(y))$ follows from the fact that $\varphi(y)$ is in the base of each P_{f_n} and that α maps the base of P_{f_n} into the base of Q_n . Therefore, $\alpha \circ \varphi = \psi \circ \alpha$.

This shows that $\alpha : \mathcal{X}_1 \rightarrow \mathcal{X}_2$ is a morphism in **SDS** and our construction shows that $\alpha^{-1}(Q_n) \leq P_{f_n}$, $n \geq 0$. Moreover, the premorphism associated to α for the sequence $(f_n)_{n=0}^\infty$ according to Definition 4.7, is obviously equivalent to f . Hence, $\mathcal{P}(\alpha) = [f]$. \square

Lemma 4.10. *The functor $\mathcal{P} : \mathbf{SDS} \rightarrow \mathbf{OBD}$ is faithful.*

Proof. Let $\mathcal{X}_1 = (X, \varphi, y, \mathcal{R})$ and $\mathcal{X}_2 = (Y, \psi, z, \mathcal{S})$ be in **SDS** and $\alpha, \beta : \mathcal{X}_1 \rightarrow \mathcal{X}_2$ be morphisms in **SDS** such that $\mathcal{P}(\alpha) = \mathcal{P}(\beta)$. We have to show that $\alpha = \beta$. Write $\mathcal{P}(\mathcal{X}_1) = (V, E)$, $\mathcal{P}(\mathcal{X}_2) = (W, S)$, $\mathcal{R} = \{P_n\}_{n=0}^\infty$, and $\mathcal{S} = \{Q_n\}_{n=0}^\infty$. Choose a sequence $(f_n)_{n=0}^\infty$, as in Definition 4.7, giving rise to an ordered premorphism $f = (F, (f_n)_{n=0}^\infty, \geq)$ for α . Similarly, choose a sequence $(g_n)_{n=0}^\infty$ giving rise to an ordered premorphism $g = (G, (g_n)_{n=0}^\infty, \geq)$ for β . Thus, since $\alpha = \beta$, f is equivalent to g .

Choose $x \in X$ and fix $n \geq 0$. There are $A_n \in P_{f_n}$ and $A'_n \in P_{g_n}$ such that $x \in A_n$ and $x \in A'_n$. Also, there are $B_n, B'_n \in Q_n$ such that $\alpha(A_n) \subseteq B_n$ and $\beta(A'_n) \subseteq B'_n$. By the version (obvious) of Proposition 2.14 for ordered premorphisms, there is $m \geq f_n, g_n$ such that $F_n \circ S_{f_n, m} \cong G_n \circ S_{g_n+1, m}$. There is $A \in P_m$ such that $x \in A$. Hence $A \subseteq A_n$ and $A \subseteq A'_n$. It is not hard to see that the relation $F_n \circ S_{f_n, m} \cong G_n \circ S_{g_n+1, m}$ implies that $B_n = B'_n$. Hence $\{\alpha(x)\} = \bigcap_{n=0}^{\infty} B_n = \bigcap_{n=0}^{\infty} B'_n = \{\beta(x)\}$. Therefore, $\alpha = \beta$. \square

We need to identify the essential range of the functor $\mathcal{P} : \mathbf{SDS} \rightarrow \mathbf{OBD}$ in order to obtain an equivalence of categories theorem. (By the essential range of a functor we mean those objects in the codomain category which are isomorphic to the objects in the range of the functor.) For an ordered Bratteli diagram (V, E, \geq) , denote by E_{\max} and E_{\min} the set of maximal and minimal edges of E , respectively. It is easy to see that there is at least one infinitely long path in E_{\max} and also in E_{\min} .

Definition 4.11 ([12], Definition 2.6). Let $B = (V, E, \geq)$ be an ordered Bratteli diagram. B is called *essentially simple* if there are unique infinitely long paths in E_{\max} and E_{\min} . That is, there is only one sequence (e_1, e_2, \dots) with each e_i in E_{\max} (E_{\min} , respectively) and $s(e_{i+1}) = r(e_i)$, for all $i \geq 1$.

Remark. Observe that essentially simple ordered Bratteli diagrams correspond to essentially minimal dynamical systems on compact, totally disconnected metrizable spaces, simple ordered Bratteli diagrams (Definition 6.1) correspond to minimal dynamical systems on compact, totally disconnected metrizable spaces (Corollary 6.3), and properly ordered Bratteli diagrams (Definition 6.6) correspond to minimal dynamical systems on the Cantor set (Corollary 6.7).

Let us denote by \mathbf{OBD}_{es} the full subcategory of \mathbf{OBD} containing all essentially simple ordered Bratteli diagrams.

Lemma 4.12. *The essential range of $\mathcal{P} : \mathbf{SDS} \rightarrow \mathbf{OBD}$ is \mathbf{OBD}_{es} .*

Proof. Let \mathcal{X} be in \mathbf{SDS} . Then the ordered Bratteli diagram $\mathcal{P}(\mathcal{X})$ is essentially simple [12, Section 4]. Now let B be an ordered Bratteli diagram which is isomorphic in \mathbf{OBD} to $\mathcal{P}(\mathcal{X})$, for some $\mathcal{X} \in \mathbf{SDS}$. By Theorem 3.6, B is equivalent to $\mathcal{P}(\mathcal{X})$. By [12, Proposition 2.7], B is also essentially simple. Hence the essential range of \mathcal{P} is contained in \mathbf{OBD}_{es} .

Let B be an essentially simple ordered Bratteli diagram. Denote by (X, φ, y) the Vershik transformation associated to B [12, Section 3]. Fix a system of Kakutani-Rokhlin partitions \mathcal{R} for (X, φ, y) (which exists by [12, Theorem 4.2]). Now by [12, Theorem 4.6], B is equivalent to $\mathcal{P}(X, \varphi, y, \mathcal{R})$. \square

Let us recall the following categorical result.

Theorem 4.13 ([13], Theorem IV.4.1). *A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is an equivalence of categories if and only if it is full, faithful, and essentially surjective, i.e., for each $d \in \mathcal{D}$ there is a $c \in \mathcal{C}$ such that $d \cong F(c)$.*

The following theorem is the main result of this section. It is a generalization of [12, Theorem 4.7] which states that there is one-to-one correspondence between equivalence classes of essentially simple ordered Bratteli diagrams and pointed topological conjugacy classes of essentially minimal t.d. dynamical systems. In the categorical language, [12, Theorem 4.7] states that the functor $\mathcal{P} : \mathbf{SDS} \rightarrow \mathbf{OBD}_{\text{es}}$, constructed above, is a classification functor (see Definition 8.5). However, it is a consequence of Theorem 4.14 that the functor $\mathcal{P} : \mathbf{SDS} \rightarrow \mathbf{OBD}_{\text{es}}$ is a strong classification functor, since each full and faithful functor is a strong classification functor [1, Lemma 5.10].

Theorem 4.14. *The functor $\mathcal{P} : \mathbf{SDS} \rightarrow \mathbf{OBD}_{\text{es}}$ is an equivalence of categories.*

Proof. Apply Theorems 4.9 and 4.13 and Lemmas 4.10 and 4.12. \square

Corollary 4.15. *Let $(X, \varphi, y, \mathcal{R})$ and $(Y, \psi, z, \mathcal{S})$ be in \mathbf{SDS} . The following statements are equivalent:*

- (1) (X, φ, y) and (Y, ψ, z) are pointed topological conjugate;
- (2) the ordered Bratteli diagrams $\mathcal{P}(X, \varphi, y, \mathcal{R})$ and $\mathcal{P}(Y, \psi, z, \mathcal{S})$ are equivalent in the sense of Herman, Putnam, and Skau;
- (3) $\mathcal{P}(X, \varphi, y, \mathcal{R})$ is isomorphic to $\mathcal{P}(Y, \psi, z, \mathcal{S})$ in the category \mathbf{OBD} .

Proof. This follows from Theorems 3.6 and 4.14. \square

5. THE FUNCTOR \mathcal{V} FROM \mathbf{OBD}_{es} TO \mathbf{SDS}

In this section we shall construct the contravariant functor $\mathcal{V} : \mathbf{OBD}_{\text{es}} \rightarrow \mathbf{SDS}$ which is the inverse of the functor $\mathcal{P} : \mathbf{SDS} \rightarrow \mathbf{OBD}_{\text{es}}$. (Note that the inverse of an equivalence of categories is unique up to natural isomorphism.) Our definition of \mathcal{V} on objects of \mathbf{OBD}_{es} coincides with the construction of [12]. We define \mathcal{V} on the morphisms in a canonical way.

Let $B = (V, E, \geq)$ be an essentially simple ordered Bratteli diagram. Denote by $\mathcal{V}(B)$ the Bratteli-Veršik dynamical system—which is an essentially minimal t.d. dynamical system—associated to B as described in [12, Section 3] and [9, Section 3]. To recall, $\mathcal{V}(B)$ is defined as follows. Let X_B denote the space of infinite paths, i.e.,

$$X_B = \{(e_1, e_2, \dots) \mid e_i \in E_i \text{ and } r(e_i) = s(e_{i+1}) \text{ for all } i \geq 1\}.$$

X_B is topologized by specifying a basis of open sets, namely the family of cylinder sets $U(e_1, e_2, \dots, e_k) = \{(f_1, f_2, \dots) \mid f_i = e_i, 1 \leq i \leq k\}$. The resulting topology is compact and Hausdorff with a countable basis of clopen sets. Each cylinder set is closed in this topology. The space X_B with this topology is called the *Bratteli compactum* associated to B . Denote by y_{\max} and y_{\min} the unique elements of E_{\max} and E_{\min} , respectively. The homeomorphism $\lambda_B : X_B \rightarrow X_B$, called the Veršik transformation, is constructed as follows. Set $\lambda_B(y_{\max}) = y_{\min}$. Let $x = (e_1, e_2, \dots) \neq y_{\max}$ and consider the smallest natural number k with $e_k \notin E_{\max}$. Let f_k denote

the successor of e_k in E (so that $r(f_k) = r(e_k)$). Let $(f_1, f_2, \dots, f_{k-1})$ denote the unique path in E_{\min} from v_0 , the unique element of V_0 , to $s(f_k)$. That is, (f_1, f_2, \dots, f_k) is the successor of (e_1, e_2, \dots, e_k) in $E_1 \circ E_2 \circ \dots \circ E_k$. Set

$$\lambda_B(e_1, e_2, \dots) = (f_1, f_2, \dots, f_k, e_{k+1}, e_{k+2}, \dots).$$

Then λ_B is a homeomorphism and $(X_B, \lambda_B, y_{\max})$ is an essentially minimal t.d. dynamical system [12, Section 3].

Let us define a canonical system of Kakutani-Rokhlin partitions $\mathcal{R}_B = \{P_n\}_{n=0}^{\infty}$ for $(X_B, \lambda_B, y_{\max})$ such that $(X_B, \lambda_B, y_{\max}, \mathcal{R}_B)$ is in **SDS**. Set $P_0 = \{X_B\}$. Fix $n \geq 1$ and define P_n as follows. For each $v \in V_n$ we have a tower T_v in P_n . For each (e_1, e_2, \dots, e_n) in $E_1 \circ E_2 \circ \dots \circ E_n$ with $r(e_n) = v$ we have an element $U(e_1, e_2, \dots, e_n)$, as defined above, in T_v . Hence,

$$P_n = \{U(e_1, e_2, \dots, e_n) \mid (e_1, e_2, \dots, e_n) \in E_1 \circ E_2 \circ \dots \circ E_n\}.$$

It is not hard to see that each P_n is a Kakutani-Rokhlin partition and that $\mathcal{R}_B = \{P_n\}_{n=0}^{\infty}$ satisfies the conditions of Definition 4.4 and hence is a system of Kakutani-Rokhlin partitions for $(X_B, \lambda_B, y_{\max})$. Finally, set $\mathcal{V}(B) = (X_B, \lambda_B, y_{\max}, \mathcal{R}_B)$. Let us summarize these facts.

Proposition 5.1. *For each ordered Bratteli diagram $B = (V, E, \geq)$, $\mathcal{V}(B) = (X_B, \lambda_B, y_{\max}, \mathcal{R}_B)$, as defined above, is in **SDS**.*

Let $B = (V, E, \geq)$ be an ordered Bratteli diagram. Define the ordered premorphism $f_B : B \rightarrow \mathcal{P}(\mathcal{V}(B))$ as follows. Set $f_B = (F_B, (n)_{n=0}^{\infty}, \geq)$ where $F_B = \{(v, T_v) \mid v \in V\}$. The decomposition of F_B is obtained by setting $F_{B,n} = \{(v, T_v) \mid v \in V_n\}$, $n \geq 0$. The source and range maps of F_B are defined by $s(v, T_v) = v$ and $r(v, T_v) = T_v$. There is only one way to define an order on F_B (since $r^{-1}\{T_v\}$ has only one element). It is not hard to see that $f_B : B \rightarrow \mathcal{P}(\mathcal{V}(B))$ is an ordered premorphism which is an isomorphism in the category of ordered Bratteli diagrams with ordered premorphisms (cf. Propositions 3.3 and 3.4). Denote by $\tau_B : B \rightarrow \mathcal{P}(\mathcal{V}(B))$ the associated ordered morphism, i.e., $\tau_B = [f_B]$, which is an isomorphism of **OBD**.

Once one fixes the isomorphisms $\tau_B = [f_B]$, then there is a unique way to define $\mathcal{V} : \mathbf{OBD}_{\text{es}} \rightarrow \mathbf{SDS}$ on morphisms to be the natural inverse of $\mathcal{P} : \mathbf{SDS} \rightarrow \mathbf{OBD}_{\text{es}}$ (see the proof of [13, Theorem IV.4.1] for details). In fact, let $h : B \rightarrow C$ be a morphism in **OBD**_{es}. Then $\tau_C h \tau_B^{-1} : \mathcal{P}(\mathcal{V}(B)) \rightarrow \mathcal{P}(\mathcal{V}(C))$ is a morphism in **OBD**_{es}. By Theorem 4.9 and Lemma 4.10, there is a unique morphism $\alpha : \mathcal{V}(C) \rightarrow \mathcal{V}(B)$ such that $\mathcal{P}(\alpha) = h$. Set $\mathcal{V}(h) = \alpha$. Thus, the following diagram commutes:

$$\begin{array}{ccc} B & \xrightarrow{\tau_B} & \mathcal{P}(\mathcal{V}(B)) \\ h \downarrow & & \downarrow \mathcal{P}(\mathcal{V}(h)) \\ C & \xrightarrow{\tau_C} & \mathcal{P}(\mathcal{V}(C)). \end{array}$$

Theorem 5.2. *The map $\mathcal{V} : \mathbf{OBD}_{\text{es}} \rightarrow \mathbf{SDS}$ as defined above, is a contravariant functor which is an equivalence of categories and the unique (up to natural isomorphism) inverse of the functor $\mathcal{P} : \mathbf{SDS} \rightarrow \mathbf{OBD}_{\text{es}}$.*

Proof. All the required properties of the map \mathcal{V} follow from the equality $\mathcal{P}(\mathcal{V}(f))\tau_B = \tau_A h$. (See the proof of [13, Theorem IV.4.1] for details.) In particular, we have $\tau : \mathbf{1}_{\mathbf{OBD}_{\text{es}}} \cong \mathcal{P}\mathcal{V}$. \square

Let us examine how the functor \mathcal{V} acts on morphisms exactly. (This is important when one intends to apply this functor.) Let $f : B \rightarrow C$ be an ordered premorphism as in Definition 3.2 and set $h = [f]$. Define a map $\alpha : X_C \rightarrow X_B$ as follows. Let $x = (s_1, s_2, \dots)$ be in X_C , i.e., an infinite path in S . Define the path $\alpha(x) = (e_1, e_2, \dots)$ in X_B as follows. Fix $n \geq 0$. By Definition 3.2, the following diagram commutes:

$$\begin{array}{ccc} V_0 & \xrightarrow{E_{0,n}} & V_n \\ F_0 \downarrow & & \downarrow F_n \\ W_0 & \xrightarrow{S_{0,f_n}} & W_{f_n}. \end{array}$$

That is, $F_0 \circ S_{0,f_n} \cong E_{0,n} \circ F_n$. Thus, there is a unique path $(e_1, e_2, \dots, e_n, d_n)$ in $E_{0,n} \circ F_n$ corresponding to the path $(s_0, s_1, \dots, s_{f_n})$ in $F_0 \circ S_{0,f_n}$, where s_0 is the unique element of F_0 . We need to check that the first n edges of the path associated to each $m > n$ are the same as those for n .

Lemma 5.3. *With the above notation, let $m > n$ and consider the path $(e'_1, e'_2, \dots, e'_m, d'_m)$ in $E_{0,m} \circ F_m$ associated to m according to the above construction. Then we have $e'_i = e_i$ for each $1 \leq i \leq n$.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccc} V_0 & \xrightarrow{E_{0,n}} & V_n & \xrightarrow{E_{n,m}} & V_m \\ F_0 \downarrow & & \downarrow F_n & & \downarrow F_m \\ W_0 & \xrightarrow{S_{0,f_n}} & W_{f_n} & \xrightarrow{S_{f_n,f_m}} & W_{f_m}. \end{array}$$

Since $(e_1, e_2, \dots, e_n, d_n)$ in $E_{0,n} \circ F_n$ is the unique path corresponding to $(s_0, s_1, \dots, s_{f_n})$ in $F_0 \circ S_{0,f_n}$, we have $r(d_n) = r(s_{f_n})$. Thus the path

$$(5.1) \quad (e_1, e_2, \dots, e_n, d_n, s_{f_n+1}, \dots, s_{f_m})$$

in $E_{0,n} \circ F_n \circ S_{f_n,f_m}$ is the unique path corresponding to $(s_0, s_1, \dots, s_{f_m})$ in $F_0 \circ S_{0,f_m}$ by the isomorphism $E_{0,n} \circ F_n \circ S_{f_n,f_m} \cong F_0 \circ S_{0,f_m}$.

On the other hand, the path $(e'_1, e'_2, \dots, e'_m, d'_m)$ in $E_{0,m} \circ F_m$ is the unique path corresponding to the path $(s_0, s_1, \dots, s_{f_m})$ by the isomorphism $E_{0,m} \circ F_m \cong F_0 \circ S_{0,f_m}$. Since the isomorphisms involved are unique (because of the order), one sees that in the isomorphism $E_{0,m} \circ F_m \cong E_{0,n} \circ F_n \circ S_{f_n,f_m}$,

the path $(e'_1, e'_2, \dots, e'_m, d'_m)$ corresponds to the path in (5.1). Therefore, $e'_i = e_i$ for each $1 \leq i \leq n$. \square

By the preceding lemma, we can define, without ambiguity, the path $\alpha(x) = (e_1, e_2, \dots)$ in X_B associated to the path $x = (s_1, s_2, \dots)$ in X_C . Thus we have a map $\alpha : X_C \rightarrow X_B$. It is not hard to see that if we replace f with another representative of the class h , then we get the same α .

Proposition 5.4. *Let $h : B \rightarrow C$ be an ordered morphism in \mathbf{OBD}_{es} and $\alpha : X_C \rightarrow X_B$ be its associated map as defined above. Then $\mathcal{V}(h) = \alpha$.*

Proof. With the notation above, it is easy to see that $\alpha(U(s_1, s_2, \dots, s_{f_n})) \subseteq U(e_1, e_2, \dots, e_n)$. This shows that α is continuous. Moreover, we have $\alpha \circ \lambda_C = \lambda_B \circ \alpha$ and α maps the unique path in S_{\max} to the unique path in E_{\max} . Thus $\alpha : \mathcal{V}(C) \rightarrow \mathcal{V}(B)$ is a morphism in \mathbf{SDS} . On the other hand, it is easily verified that $\mathcal{P}(\alpha)\tau_B = \tau_A h$. Hence $\mathcal{P}(\alpha) = \mathcal{P}(\mathcal{V}(h))$ and by Lemma 4.10, we get $\mathcal{V}(h) = \alpha$. \square

As stated in the proof of Theorem 5.2, the correspondence τ , as defined above, gives $1_{\mathbf{OBD}_{\text{es}}} \cong \mathcal{PV}$. Using this, a standard categorical procedure gives a correspondence σ which implements $1_{\mathbf{SDS}} \cong \mathcal{VP}$. In fact, let \mathcal{X} be in \mathbf{SDS} . Thus, $\tau_{\mathcal{P}(\mathcal{X})} : \mathcal{P}(\mathcal{X}) \rightarrow \mathcal{P}(\mathcal{V}(\mathcal{P}(\mathcal{X})))$ is an isomorphism in \mathbf{OBD} . By Theorem 4.9 and Lemma 4.10, there is a unique isomorphism $\sigma_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{V}(\mathcal{P}(\mathcal{X}))$ such that $\mathcal{P}(\sigma_{\mathcal{X}}) = \tau_{\mathcal{P}(\mathcal{X})}^{-1}$. Moreover, we have $\sigma : 1_{\mathbf{SDS}} \cong \mathcal{VP}$. Let us describe how σ acts exactly.

Let $\mathcal{X} = (X, \varphi, y, \mathcal{R})$ be in \mathbf{SDS} where $\mathcal{R} = \{P_n\}_{n=0}^{\infty}$. Choose $x \in X$. For each $n \geq 0$, let S_n denote the tower of P_n containing x . Fix $n \geq 1$ and denote by k_n the height at which S_n passes through S_{n-1} (see Definition 4.7). Set $e_n = (n, S_{n-1}, S_n, k_n)$ and $\alpha(x) = (e_1, e_2, \dots)$. Thus we have a map $\alpha : X \rightarrow X_B$ where $B = (V, E) = \mathcal{P}(\mathcal{X})$.

Lemma 5.5. *With the above notation, we have $\sigma_{\mathcal{X}} = \alpha$.*

Proof. First we show that $\alpha : \mathcal{X} \rightarrow \mathcal{V}(\mathcal{P}(\mathcal{X}))$ is a morphism in \mathbf{SDS} . To this end, we need to show that α is continuous, $\alpha(y) = y_{\max}$, and $\alpha \circ \varphi = \lambda_B \circ \alpha$. Let $x \in X$ and $\alpha(x) = (e_1, e_2, \dots)$ be as above. For each $n \geq 1$, let A_n denote the element of the tower S_n such that $x \in A_n$. Then we have $A_1 \supseteq A_2 \supseteq \dots$. It follows that $\alpha(A_n) \subseteq U(e_1, e_2, \dots, e_n)$, $n \geq 1$. This proves that α is continuous. The equalities $\alpha(y) = y_{\max}$ and $\alpha \circ \varphi(y) = \lambda_B \circ \alpha(y)$ are easy to check. Let $x \in X \setminus \{y\}$ and k be the smallest natural number such that x is not in the top of P_k . Let $\alpha(\varphi(x)) = (f_1, f_2, \dots)$. Then, for each $n \geq k$, both x and $\varphi(x)$ are in the same tower of P_n (i.e., the tower S_n). Hence $f_n = e_n$, for each $n \geq k + 1$. On the other hand, for each $1 \leq n \leq k - 1$, since x is in the top of P_n , $\varphi(x)$ is in the bottom of P_n . This shows that (f_1, f_2, \dots, f_k) is the successor of (e_1, e_2, \dots, e_k) in $E_1 \circ E_2 \circ \dots \circ E_k$. Therefore, $\alpha(\varphi(x)) = \lambda_B(\alpha(x))$.

To see that $\sigma_{\mathcal{X}} = \alpha$ it is enough to show that $\tau_{\mathcal{P}(\mathcal{X})}\mathcal{P}(\alpha) = \text{id}_C$ where $C = \mathcal{P}(\mathcal{V}(\mathcal{P}(\mathcal{X})))$ (since \mathcal{P} is faithful). Observe that α is bijective and with

the above notation, $\alpha(A_n) = U(e_1, e_2, \dots, e_n)$, $n \geq 1$. Let $\mathcal{R}_B = \{Q_n\}_{n=0}^\infty$ denote the scale of the system $\mathcal{V}(\mathcal{P}(\mathcal{X}))$. Then we have $\alpha(P_n) = Q_n$, $n \geq 0$. Thus in applying Definition 4.7 to obtain $\mathcal{P}(\alpha)$ we can take $f_n = n$, $n \geq 0$. Using this it is not hard to see that $\tau_{\mathcal{P}(\mathcal{X})}\mathcal{P}(\alpha) = \text{id}_C$. \square

Let us summarize the results of Sections 4 and 5.

Theorem 5.6. *The contravariant functors $\mathcal{P} : \mathbf{SDS} \rightarrow \mathbf{OBD}_{\text{es}}$ and $\mathcal{V} : \mathbf{OBD}_{\text{es}} \rightarrow \mathbf{SDS}$ are equivalence of categories which are inverse of each other and $\tau : 1_{\mathbf{OBD}_{\text{es}}} \cong \mathcal{P}\mathcal{V}$ and $\sigma : 1_{\mathbf{SDS}} \cong \mathcal{V}\mathcal{P}$.*

6. CANTOR MINIMAL SYSTEMS

In this section we apply the results of Sections 4 and 5 to Cantor minimal dynamical systems. First, we investigate the minimal dynamical systems on compact, totally disconnected metrizable spaces (Corollary 6.3). Then we will deal with minimal dynamical systems on Cantor sets (i.e., compact, totally disconnected metrizable spaces with no isolated points—which we know are all homeomorphic—if they are non-empty!).

Definition 6.1 ([4], Section 2.1). Let $B = (V, E)$ be a Bratteli diagram (as in Definition 2.1). B is called *simple* if there exists a telescoping (V', E') of (V, E) such that the embedding matrices of (V', E') have only non-zero elements in each level. In other words, B is simple if for each $n \geq 0$ there is $m > n$ such that for every $v \in V_n$ and for every $w \in V_m$ there is a path in $E_{n,m}$ from v to w .

The following is part of the literature (it follows from the results of [12]).

Proposition 6.2. *Let $B = (V, E, \geq)$ be an essentially simple ordered Bratteli diagram. Then the following statements are equivalent:*

- (1) *the system (X_B, λ_B) is minimal;*
- (2) *(V, E) is a simple Bratteli diagram.*

Denote by $\mathbf{OBD}_{\text{ess}}$ the full subcategory of \mathbf{OBD}_{es} consisting of essentially simple ordered Bratteli diagrams which are simple. Also, denote by \mathbf{SDS}_{m} the full subcategory of \mathbf{SDS} consisting of scaled minimal dynamical systems on compact, totally disconnected metrizable spaces. The following statement follows from Theorem 5.6.

Corollary 6.3. *The categories \mathbf{SDS}_{m} and $\mathbf{OBD}_{\text{ess}}$, as defined above, are equivalent. More precisely, the contravariant functors $\mathcal{P} : \mathbf{SDS}_{\text{m}} \rightarrow \mathbf{OBD}_{\text{ess}}$ and $\mathcal{V} : \mathbf{OBD}_{\text{ess}} \rightarrow \mathbf{SDS}_{\text{m}}$ are inverse of each other.*

Recall that for a Bratteli diagram $B = (V, E)$, the Bratteli compactum X_B (defined at the beginning of Section 5) is a compact, totally disconnected metrizable space. Thus, to obtain a Cantor set we need only consider the property of having no isolated points and its meaning on the diagram B .

Lemma 6.4. *Let $B = (V, E)$ be a Bratteli diagram. Then the following statements are equivalent:*

- (1) X_B is homeomorphic to the Cantor set;
- (2) for each infinite path $x = (e_1, e_2, \dots)$ in X_B and each $n \geq 1$ there is an infinite path $y = (f_1, f_2, \dots)$ with $x \neq y$ and $e_k = f_k$, $1 \leq k \leq n$;
- (3) for each $n \geq 0$ and each $v \in V_n$ there is $m \geq n$ and $w \in V_m$ such that there is path from v to w and $|s^{-1}(\{w\})| \geq 2$.

Proof. In view of the definition of the topology of X_B (Section 5), (2) is equivalent to having no isolated points. Thus (1) is equivalent to (2). Also, it is easy to see that (3) is just a reformulation of (2). \square

Proposition 6.5. *Let $B = (V, E)$ be a simple Bratteli diagram. Then the following statements are equivalent:*

- (1) X_B is homeomorphic to the Cantor set;
- (2) X_B is infinite;
- (3) the set $\{n \in \mathbb{N} \mid |E_n| \geq 2\}$ is infinite.

Proof. It is immediate that (1) implies (2) and also (2) implies (3). Moreover, it is not hard to see that (3) and the fact that B is simple imply item (3) of Lemma 6.4. \square

Definition 6.6 ([4]). Let $B = (V, E, \geq)$ be an essentially simple ordered Bratteli diagram. B is called *properly ordered* if it is simple and X_B is infinite (cf. Proposition 6.5).

Denote by \mathbf{OBD}_{po} the full subcategory of \mathbf{OBD}_{es} consisting of properly ordered Bratteli diagrams. Also, denote by \mathbf{SDS}_{cm} the full subcategory of \mathbf{SDS} consisting of scaled minimal dynamical systems on Cantor sets. The following statement follows from Theorem 5.6.

Corollary 6.7. *The categories \mathbf{SDS}_{cm} and \mathbf{OBD}_{po} , as defined above, are equivalent. More precisely, the contravariant functors $\mathcal{P} : \mathbf{SDS}_{\text{cm}} \rightarrow \mathbf{OBD}_{\text{po}}$ and $\mathcal{V} : \mathbf{OBD}_{\text{po}} \rightarrow \mathbf{SDS}_{\text{cm}}$ are inverse of each other.*

7. THE DIRECT LIMIT FUNCTOR

In this section we define the functor $\mathcal{D} : \mathbf{BD} \rightarrow \mathbf{DG}$ from the category of Bratteli diagrams to the category of scaled dimension groups. It is well known that one can associate to each Bratteli diagram B a dimension group, say $\mathcal{D}(B)$ [7, 12]. The nontrivial part is how to define the map \mathcal{D} on morphisms of \mathbf{BD} in such a way that $\mathcal{D} : \mathbf{BD} \rightarrow \mathbf{DG}$ is a functor, and in fact an equivalence of categories. Although it is already known that the category of Bratteli diagrams and the category of scaled dimension groups are equivalent ([1, Corollary 6.5]), we give an explicit functor giving this equivalence. (The idea of the construction of this functor was mentioned in the proof of [1, Corollary 6.5].)

Let us recall some definitions about dimension groups. By an *ordered group* we mean a pair (G, G^+) where G is an abelian group and G^+ is a (positive) cone for G , i.e., $G^+ + G^+ \subseteq G^+$, $G^+ - G^+ = G$, and $G^+ \cap (-G^+) = \{0\}$. The partial order induced by G^+ on G is defined by $x \geq y$ if $x - y \in G^+$.

By an *order unit* for (G, G^+) we mean an element u in G^+ such that, for every g in G^+ , $nu \geq g$ for some n in \mathbb{N} . For each $m \in \mathbb{N}$, the standard cone of the group \mathbb{Z}^m is defined by $(\mathbb{Z}^m)^+ = \{(n_1, \dots, n_m) \mid n_i \geq 0, 1 \leq i \leq m\}$.

A *dimension group* is a countable ordered group obtained as a direct limit of a sequence of finitely generated free abelian groups with standard order and positive group homomorphisms as maps [7]. It is known that a countable ordered group (G, G^+) is a dimension group if, and only if, it is unperforated (i.e., if $na \geq 0$ for some $n \in \mathbb{N}$, then $a \geq 0$, $a \in G$) and has the Riesz interpolation property, i.e., if $a_1, a_2, b_1, b_2 \in G$ with $a_i \leq b_j$, $1 \leq i, j \leq 2$, then there exists $c \in G$ with $a_i \leq c \leq b_j$, $1 \leq i, j \leq 2$ [6].

A *scale* for an ordered group (G, G^+) is a subset Γ of G^+ which is generating, hereditary, and directed [5]. Let us denote by **DG** the category of scaled dimension groups (i.e., triples of the form (G, G^+, Γ) where (G, G^+) is a dimension group and Γ is a scale) with order and scale preserving homomorphisms (see [7, 6, 5, 20]).

Definition 7.1. Let $B = (V, E)$ be in **BD** (i.e., a Bratteli diagram as in Definition 2.2) and let us define a scaled dimension group $\mathcal{D}(B) = (G, G^+, \Gamma)$ as follows. Write $V = (V_n)_{n=1}^\infty$, $E = (E_n)_{n=1}^\infty$, and $V_n^T = (m_{n1} \ m_{n2} \ \dots \ m_{nk_n})$, $n \geq 1$. Set $G_n = \mathbb{Z}^{k_n}$ and let G_n^+ denote its standard cone. Set

$$\Gamma_n = \{(a_1, a_2, \dots, a_{k_n}) \in G_n \mid 0 \leq a_i \leq m_{ni}, 1 \leq i \leq k_n\}.$$

Thus, (G_n, G_n^+, Γ_n) is a scaled dimension group. Let $\varphi_n : G_n \rightarrow G_{n+1}$ be the homomorphism defined by the multiplication by E_n . Observe that $\varphi_n(\Gamma_n) \subseteq \Gamma_{n+1}$, since $E_n V_n \leq V_{n+1}$. Set $\mathcal{D}(B) = \varinjlim (G_n, \varphi_n)$. More precisely, $\mathcal{D}(B)$ is defined as follows. Set

$$G_\infty = \left\{ (a_n) \in \prod_{n=1}^\infty G_n \mid \exists m \geq 1 \ a_{n+1} = \varphi_n(a_n) \ (n \geq m) \right\},$$

and let G_0 be the subgroup of G_∞ consisting of sequences which are zero everywhere except in finitely many entries. Set $G = G_\infty / G_0$. Moreover, let $\pi : G_\infty \rightarrow G$ be the quotient map and set $G^+ = \pi(G_\infty \cap \prod_{n=1}^\infty G_n^+)$ and $\Gamma = \pi(G_\infty \cap \prod_{n=1}^\infty \Gamma_n)$. Set $\mathcal{D}(B) = (G, G^+, \Gamma)$ which is a scaled dimension group and so is in **DG**.

Definition 7.2. Let $B, C \in \mathbf{BD}$ and $h : B \rightarrow C$ be a morphism in **BD** (see [1, Section 2] for definition). Let us define a morphism $\mathcal{D}(h) : \mathcal{D}(B) \rightarrow \mathcal{D}(C)$ in **DG** as follows. Let f be a premorphism in **BD** such that $h = [f]$. Write $B = (V, E)$, $C = (W, S)$, and $f = ((F_n)_{n=1}^\infty, (f_n)_{n=1}^\infty)$. Let $\mathcal{D}(B) = (G, G^+, \Gamma)$ and $\mathcal{D}(C) = (H, H^+, \Lambda)$ be as in Definition 7.1, with connecting maps $(\varphi_n)_{n=1}^\infty$ and $(\psi_n)_{n=1}^\infty$, respectively. Let $\eta_n : G_n \rightarrow H_{f_n}$ denote the homomorphism defined by the multiplication by F_n . Thus we have the following commutative diagram:

$$\begin{array}{ccccccc}
G_1 & \xrightarrow{\varphi_1} & G_2 & \xrightarrow{\varphi_2} & G_3 & \xrightarrow{\varphi_3} & \cdots \\
\eta_1 \downarrow & & \eta_2 \downarrow & \swarrow \eta_3 & & & \\
H_1 & \xrightarrow{\psi_1} & H_2 & \xrightarrow{\psi_2} & H_3 & \xrightarrow{\psi_3} & \cdots
\end{array}$$

Therefore, we obtain a group homomorphism $\eta : G \rightarrow H$ with $\eta(G^+) \subseteq H^+$ and $\eta(\Gamma) \subseteq \Lambda$. More precisely, for each $(a_n)_{n=1}^\infty$ in G_∞ let $\eta((a_n)_{n=1}^\infty + G_0) = ((b_n)_{n=1}^\infty + H_0)$ be defined as follows. Let m be a natural number such that $a_{n+1} = \varphi_n(a_n)$, $n \geq m$. Set $b_1 = b_2 = \cdots = b_{f_m-1} = 0$, $b_{f_n} = \eta_n(a_n)$ for $n \geq m$, and $b_k = \varphi_{k-1}\varphi_{k-2}\cdots\varphi_{f_n}(b_{f_n})$ for $n \geq m$ and $f_n < k < f_{n+1}$. It is not hard to see that η has the stated properties. Note that η does not depend on the representative f of the equivalence class h (see the proof of Proposition 7.3 below). Now set $\mathcal{D}(h) = \eta$ which is a morphism from $\mathcal{D}(B)$ to $\mathcal{D}(C)$ in the category **DG**.

Proposition 7.3. *The map $\mathcal{D} : \mathbf{BD} \rightarrow \mathbf{DG}$, as defined above, is a functor.*

Proof. First note that, with the notation of Definition 7.2, $\mathcal{D}(h)$ does not depend on the representative f of the equivalence class h . This fact follows easily from [1, Proposition 2.11]. That \mathcal{D} preserves composition and identity is also easy to check. Therefore, \mathcal{D} is a functor. \square

Theorem 7.4. *The functor $\mathcal{D} : \mathbf{BD} \rightarrow \mathbf{DG}$ is an equivalence of categories.*

Proof. We will show that the functor \mathcal{D} is full, faithful, and essentially surjective. Then the statement follows from Theorem 4.13. First let us show that \mathcal{D} is faithful. Let $B, C \in \mathbf{BD}$ and $h, k : B \rightarrow C$ be morphisms in **BD** such that $\mathcal{D}(h) = \mathcal{D}(k)$. Let $f = ((F_n)_{n=1}^\infty, (f_n)_{n=1}^\infty)$ and $g = ((K_n)_{n=1}^\infty, (g_n)_{n=1}^\infty)$ be premorphisms such that $h = [f]$ and $k = [g]$. We have to show that f is equivalent to g .

Write $B = (V, E)$ and $C = (W, S)$. Let $\mathcal{D}(B) = (G, G^+, \Gamma)$ and $\mathcal{D}(C) = (H, H^+, \Lambda)$ as in Definition 7.1, with connecting maps $(\varphi_n)_{n=1}^\infty$ and $(\psi_n)_{n=1}^\infty$, respectively. Fix $n \geq 1$ and let $m \geq 1$ be such that $G_n = \mathbb{Z}^m$. Moreover, let $\{e_1, e_2, \dots, e_m\}$ denote the standard basis of G_n (as a \mathbb{Z} -module). Define $x_1, x_2, \dots, x_m \in G_\infty$ as follows. Write $x_i = (x_{ik})_{k=1}^\infty$ such that $x_{i1} = x_{i2} = \cdots = x_{i(n-1)} = 0$, $x_{in} = e_i$, and $x_{ik} = \varphi_{n,k}(e_i)$ for $k > n$, where $\varphi_{n,k} = \varphi_{k-1}\varphi_{k-2}\cdots\varphi_n$ which is a homomorphism from G_n to G_k . Let $\eta_n : G_n \rightarrow H_{f_n}$ and $\theta_n : G_n \rightarrow H_{g_n}$ be the maps obtained by applying Definition 7.2 to f and g , respectively. Since $\mathcal{D}(h)(x_i + G_0) = \mathcal{D}(k)(x_i + G_0)$, $1 \leq i \leq m$, there is $l \geq f_n, g_n$ such that $\psi_{f_n,l} \circ \eta_n(e_i) = \psi_{g_n,l} \circ \theta_n(e_i)$, $1 \leq i \leq m$. Thus, $\psi_{f_n,l} \circ \eta_n = \psi_{g_n,l} \circ \theta_n$ and hence $S_{f_n l} F_n = S_{g_n l} K_n$. By [1, Proposition 2.11], f is equivalent to g . Therefore, \mathcal{D} is faithful.

Next let us show that \mathcal{D} is full. In fact, the proof is based on a one-sided intertwining argument. Let $B, C \in \mathbf{BD}$ and $\varphi : \mathcal{D}(B) \rightarrow \mathcal{D}(C)$ be a morphism in **DG**. We have to show that there is a morphism $h : B \rightarrow C$ in **BD** such that $\mathcal{D}(h) = \varphi$. We retain the notation of the preceding paragraph.

Let $\varphi^n : G_n \rightarrow G$ and $\psi^n : H_n \rightarrow H$, $n \geq 1$, be the homomorphisms coming from the construction of the direct limit in Definition 7.1. Then we have $\Gamma = \bigcup_{n=1}^{\infty} \varphi^n(\Gamma_n)$ and $\Lambda = \bigcup_{n=1}^{\infty} \psi^n(\Lambda_n)$. Since each G_n is finitely generated, there is a strictly increasing sequence $(f_n)_{n=1}^{\infty}$ in \mathbb{N} such that $\varphi(\varphi^n(\Gamma_n)) \subseteq \psi^{f_n}(\Lambda_{f_n})$, $n \geq 1$. Thus for each $n \geq 1$, there is a (not necessarily unique) order and scale preserving homomorphism $\theta_n : G_n \rightarrow H_{f_n}$ such that $\varphi\varphi^n = \psi^{f_n}\theta_n$. Consider the following (possibly non-commutative) diagram:

$$\begin{array}{ccccc} G_n & \xrightarrow{\varphi_n} & G_{n+1} & \xrightarrow{\varphi^{n+1}} & G \\ \theta_n \downarrow & & \downarrow \theta_{n+1} & & \downarrow \varphi \\ H_{f_n} & \xrightarrow{\psi_{f_n, f_{n+1}}} & H_{f_{n+1}} & \xrightarrow{\psi_{f_{n+1}}} & H \end{array}$$

The right hand side square commutes. Also, if we omit θ_{n+1} the remaining diagram commutes. Thus, $\psi^{f_{n+1}}\theta_{n+1}\varphi_n = \psi^{f_{n+1}}\psi_{f_n, f_{n+1}}\theta_n$. Again since G_n is finitely generated, there is $m_n \geq f_{n+1}$ such that $\psi_{f_{n+1}, m_n}\theta_{n+1}\varphi_n = \psi_{f_{n+1}, m_n}\psi_{f_n, f_{n+1}}\theta_n = \psi_{f_n, m_n}\theta_n$ as maps from G_n to H_{m_n} . This shows that we can (so let us) replace the sequence $(f_n)_{n=1}^{\infty}$ with another sequence (denoted with the same notation) and the maps θ_n 's with η_n 's such that the left hand side square in the above diagram commutes (for each $n \geq 1$). Thus we obtain a strictly increasing sequence $(f_n)_{n=1}^{\infty}$ and order and scale preserving homomorphisms $\eta_n : G_n \rightarrow H_{f_n}$, $n \geq 1$, such that $\varphi\varphi^n = \psi^{f_n}\eta_n$ and $\eta_{n+1}\varphi_n = \psi_{f_n, f_{n+1}}\eta_n$. For each $n \geq 1$, there is a unique matrix F_n of positive (i.e., non-negative) integers such that η_n is multiplication by F_n . Set $f = ((F_n)_{n=1}^{\infty}, (f_n)_{n=1}^{\infty})$. Now it is easy to check that f is a premorphism in **BD** and $\mathcal{D}([f]) = \varphi$. Hence \mathcal{D} is a full functor.

Finally, that \mathcal{D} is essentially surjective follows from the Effros-Handelman-Shen theorem [6, Theorem 2.2]. \square

Proposition 7.5. *The following diagram of functors commutes (up to natural isomorphism):*

$$\begin{array}{ccc} \overline{\mathbf{AF}^{\text{out}}} & \xrightarrow{\overline{\mathbf{B}}} & \mathbf{BD} \\ & \searrow \overline{\mathbf{K}}_0 & \downarrow \mathcal{D} \\ & & \mathbf{DG} \end{array}$$

Moreover, all three functors in this diagram are equivalence of categories.

Proof. By Theorem 7.4 and [1, Theorems 5.11 and 6.4], all three functors in the above diagram are equivalence of categories. That the diagram commutes (up to natural isomorphism) follows from the continuity of the functor \mathbf{K}_0 with respect to direct limits. \square

8. THE FUNCTOR K^0

In this section we investigate the functor $K^0 : \mathbf{DS} \rightarrow \mathbf{DG}_1$ from the category of essentially minimal t.d. dynamical systems on totally disconnected, compact metrizable spaces (Definition 4.2) to the category of unital dimension groups (Definition 8.2). Let X be a totally disconnected, compact metrizable space and let φ be a homeomorphism of X . Let $C(X, \mathbb{Z})$ denote the group of continuous functions from X to \mathbb{Z} . Set

$$C(X, \mathbb{Z})^+ = \{f \in C(X, \mathbb{Z}) \mid f \geq 0\}.$$

Let $\mathbf{1}$ be the constant function with value 1 which is an order unit for $C(X, \mathbb{Z})$. Denote by φ_* the automorphism of $C(X, \mathbb{Z})$ sending f to $f \circ \varphi^{-1}$. Then $\text{id} - \varphi_*$ is an endomorphism of $C(X, \mathbb{Z})$, where id is the identity map. Define $K^0(X, \varphi)$ as in [12], i.e.,

$$K^0(X, \varphi) = \frac{C(X, \mathbb{Z})}{B_\varphi},$$

where B_φ denotes the subgroup $\text{Im}(\text{id} - \varphi_*)$ of $C(X, \mathbb{Z})$.

Proposition 8.1 ([12]). *Let (X, φ, y) be an essentially minimal t.d. system. If we define $K^0(X, \varphi)^+$ to be the image of $C(X, \mathbb{Z})^+$ in the quotient group $K^0(X, \varphi)$ and if we also let $\mathbf{1}$ denote the image of $\mathbf{1}$, then*

$$(K^0(X, \varphi), K^0(X, \varphi)^+)$$

is a dimension group and $\mathbf{1}$ is an order unit.

Definition 8.2. Denote by \mathbf{DG}_1 the category whose objects are triples (G, G^+, u) where (G, G^+) is a dimension group and u is a fixed order unit. A morphism in \mathbf{DG}_1 is a positive homomorphism preserving the distinguished order units. Let us refer to \mathbf{DG}_1 as the category of unital dimension groups.

Definition 8.3. Define the contravariant functor $K^0 : \mathbf{DS} \rightarrow \mathbf{DG}_1$ as follows. For each essentially minimal t.d. dynamical system $\mathcal{X} = (X, \varphi, y)$ set $K^0(\mathcal{X}) = (K^0(X, \varphi), K^0(X, \varphi)^+, \mathbf{1})$. The definition K^0 on morphisms proceeds in a natural way. In fact, let $\alpha : (X, \varphi, y) \rightarrow (Y, \psi, z)$ be a morphism in \mathbf{DS} . Consider the group homomorphism $C(\alpha) : C(Y, \mathbb{Z}) \rightarrow C(X, \mathbb{Z})$ defined by $C(\alpha)(f) = f \circ \alpha$. It is easy to see that $C(\alpha)(B_\psi) \subseteq \text{Im}(B_\varphi)$. Set

$$K^0(\alpha)(f + B_\psi) = f \circ \alpha + B_\varphi.$$

The proof of the following statement is straightforward.

Proposition 8.4. *The map $K^0 : \mathbf{DS} \rightarrow \mathbf{DG}_1$, as defined above, is a contravariant functor.*

Let us recall the notion of a (strong) classification functor.

Definition 8.5 (Elliott, [8]). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called a *classification functor* if $F(a) \cong F(b)$ implies $a \cong b$, for each $a, b \in \mathcal{C}$; it is called a *strong classification functor* if each isomorphism from $F(a)$ to $F(b)$ is the image of

an isomorphism from a to b . Thus in this case we have $a \cong b$ if and only if $F(a) \cong F(b)$.

Remark. The functor $K^0 : \mathbf{DS} \rightarrow \mathbf{DG}_1$ is not a (strong) classification functor. In fact, in view of Theorem 9.4, we need a weaker notion of morphism between dynamical systems to obtain a classification functor. However, the existence of this functor may be important from other points of view. For example, Glasner and Weiss used this (for the special case of Cantor systems) to deal with their weak orbit equivalence (see [10, Proposition 3.1]).

9. THE FUNCTOR \mathcal{AF} FROM \mathbf{DS} TO \mathbf{AF}

In this section we construct the functor \mathcal{AF} from the category of essentially minimal t.d. dynamical systems (Definition 4.2) to the category of AF algebras.

Let (X, φ, y) be an essentially minimal t.d. dynamical system (Definition 4.1) and $C(X) \rtimes_{\varphi} \mathbb{Z}$ denote the associated crossed product C^* -algebra as described in [12]. Set $C(X) \rtimes_{\varphi} \mathbb{Z} = \mathcal{A}(X, \varphi, y)$. In fact, the homeomorphism φ of X induces the automorphism $C(\varphi^{-1})$ of $C(X)$ defined by $C(\varphi^{-1})(f) = f \circ \varphi^{-1}$, $f \in C(X)$. Then \mathbb{Z} acts on $C(X)$ by means of this automorphism and one considers the resulting crossed product C^* -algebra $C(X) \rtimes_{\varphi} \mathbb{Z}$. See [15], [3, Chapter VIII], and [21] for the definition of (discrete) crossed products of C^* -algebras. Also, each morphism in \mathbf{DS} induces a morphism in the category of unital C^* -algebras with unital $*$ -homomorphisms, \mathcal{C}_1^* (see the proof of the following proposition).

Proposition 9.1. *The map $\mathcal{A} : \mathbf{DS} \rightarrow \mathcal{C}_1^*$, as defined above, is a contravariant functor.*

Proof. Let (X, φ, y) and (Y, ψ, z) be in \mathbf{DS} and $\alpha : (X, \varphi, y) \rightarrow (Y, \psi, z)$ be a morphism, i.e., a homomorphism from the dynamical system (X, φ) to (Y, ψ) (a continuous map with $\alpha \circ \varphi = \psi \circ \alpha$) such that $\varphi(y) = z$. Then $C(\alpha) : C(Y) \rightarrow C(X)$ defined by $C(\alpha)(f) = f \circ \alpha$, is an equivariant homomorphism. In fact, for each $n \in \mathbb{Z}$ and each $f \in C(Y)$ we have

$$C(\alpha)(f \circ \psi^n) = f \circ \psi^n \circ \alpha = f \circ \alpha \circ \varphi^n = C(\alpha)(f) \circ \varphi^n.$$

Therefore, by [21, Corollary 2.48], $C(\alpha)$ induces a unital $*$ -homomorphism $\mathcal{A}(\alpha) : C(Y) \rtimes_{\psi} \mathbb{Z} \rightarrow C(X) \rtimes_{\varphi} \mathbb{Z}$ such that $\mathcal{A}(\alpha)(g) = C(\alpha) \circ g$, for each $g \in C_c(\mathbb{Z}, C(Y))$. It is easy to check that the map $\mathcal{A} : \mathbf{DS} \rightarrow \mathcal{C}_1^*$ preserves identities and composition and so is a contravariant functor. \square

Remark. In the proof of the above proposition, we did not use the essentially minimality. In fact, the contravariant functor \mathcal{A} may also be defined for general dynamical systems.

Definition 9.2. Let X be a totally disconnected metrizable compact space, and let $\mathcal{X} = (X, \varphi, y)$ be an essentially minimal t.d. dynamical system. Denote by $\mathcal{AF}(\mathcal{X})$ the C^* -subalgebra of $\mathcal{A}(\mathcal{X}) = C(X) \rtimes_{\varphi} \mathbb{Z}$ generated by

$C(X)$ and $u \cdot C_0(X \setminus \{y\})$ where u is the canonical unitary determining φ in $C(X) \rtimes_{\varphi} \mathbb{Z}$. By [17, Theorem 3.3], $\mathcal{AF}(\mathcal{X})$ is an AF algebra.

Remark. The definition of an AF algebra associated with an essentially minimal t.d. dynamical system (X, φ, y) has been taken from [17, 12]. In [12], the authors use the notation $AF(X, \varphi, y)$. Also, for each closed subset Z of X , the C^* -subalgebra of $\mathcal{A}(\mathcal{X}) = C(X) \rtimes_{\varphi} \mathbb{Z}$ generated by $C(X)$ and $u \cdot C_0(X \setminus Z)$ is an AF algebra [17].

The functor \mathcal{A} in Proposition 9.1 preserves the AF algebra parts and hence we obtain a functor \mathcal{AF} from **DS** to the category of AF algebras, **AF** (see the proof of the following proposition).

Proposition 9.3. *The map $\mathcal{AF} : \mathbf{DS} \rightarrow \mathbf{AF}$, as defined above, is a contravariant functor.*

Proof. Let us examine the definition of the map \mathcal{AF} on morphisms. Let $\mathcal{X}_1 = (X, \varphi, y)$ and $\mathcal{X}_2 = (Y, \psi, z)$ be in **DS** and $\alpha : \mathcal{X}_1 \rightarrow \mathcal{X}_2$ be a morphism in **DS**. Consider the $*$ -homomorphism $\mathcal{A}(\alpha) : \mathcal{A}(\mathcal{X}_2) \rightarrow \mathcal{A}(\mathcal{X}_1)$ as in Proposition 9.1. We show that $\mathcal{A}(\alpha)(\mathcal{AF}(\mathcal{X}_2)) \subseteq \mathcal{AF}(\mathcal{X}_1)$. It is obvious that $\mathcal{A}(\alpha)(C(Y)) \subseteq C(X)$. Denote by u_1 and u_2 the canonical unitaries determining φ and ψ in $C(X) \rtimes_{\varphi} \mathbb{Z}$ and $C(Y) \rtimes_{\psi} \mathbb{Z}$, respectively. Then we have $\mathcal{A}(\alpha)(u_2) = u_1$. Now let $f \in C_0(Y \setminus \{z\})$. Then,

$$\mathcal{A}(\alpha)(f)(y) = (f \circ \alpha)(y) = f(\alpha(y)) = f(z) = 0.$$

This shows that $\mathcal{A}(\alpha)(C_0(Y \setminus \{z\})) \subseteq C_0(X \setminus \{y\})$. Therefore, in view of Definition 9.2, $\mathcal{A}(\alpha)(\mathcal{AF}(\mathcal{X}_2)) \subseteq \mathcal{AF}(\mathcal{X}_1)$. Denote by $\mathcal{AF}(\alpha) : \mathcal{AF}(\mathcal{X}_2) \rightarrow \mathcal{AF}(\mathcal{X}_1)$ the restriction of $\mathcal{A}(\alpha)$ to $\mathcal{AF}(\mathcal{X}_2)$. By Proposition 9.1, $\mathcal{AF} : \mathbf{DS} \rightarrow \mathbf{AF}$ thus defined is a contravariant functor. \square

Remark. The functor $\mathcal{AF} : \mathbf{DS} \rightarrow \mathbf{AF}$ is faithful. This is because for each morphism $\alpha : \mathcal{X}_1 \rightarrow \mathcal{X}_2$ as in the proof of Proposition 9.3, we have $\mathcal{AF}(\alpha)(C(Y)) = \mathcal{A}(\alpha)(C(Y)) \subseteq C(X)$ and $\mathcal{AF}(\alpha)(f) = f \circ \alpha$, $f \in C(Y)$. The functor \mathcal{AF} is not a strong classification functor (by Theorem 9.4 below). Therefore, \mathcal{AF} is not a full functor, since each full and faithful functor is a strong classification functor (by [1, Lemma 5.10]). Similarly, the functor $\mathcal{A} : \mathbf{DS} \rightarrow \mathcal{C}_1^*$ is faithful but not full. Also, it is not a strong classification functor.

That the first three statements in the following theorem are equivalent is a well-known result [9]. Recall that a minimal dynamical system (X, φ) is called a *Cantor minimal system* if X is a compact metrizable space with a countable basis of clopen subsets and X has no isolated points (i.e., X is homeomorphic to the Cantor set). See [9] for the definition of the notion of strong orbit equivalence.

Theorem 9.4. *Let (X, φ) and (Y, ψ) be Cantor minimal systems. Let y and z be arbitrary points in X and Y , respectively. Then the following statements are equivalent:*

- (1) (X, φ) and (Y, ψ) are strong orbit equivalent;
- (2) $K^0(X, \varphi)$ is order isomorphic to $K^0(Y, \psi)$ by a map preserving the distinguished order unit;
- (3) $C(X) \rtimes_{\varphi} \mathbb{Z} \cong C(Y) \rtimes_{\psi} \mathbb{Z}$;
- (4) $\mathcal{AF}(X, \varphi, y) \cong \mathcal{AF}(Y, \psi, z)$ in **AF**.

Proof. The theorem follows from [9, Theorem 2.1] and [12, Theorem 5.3]. \square

Remark. One way to obtain a functorial formulation of the preceding theorem is to define a suitable notion of morphism between Cantor minimal systems such that isomorphism coincides with strong orbit equivalence as introduced in [9]. This needs more investigation.

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